Lesson 16: Log-binomial Regression

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Learning Objectives

1. Revisit the key distinctions between logistic and log-binomial regression.
2. Interpret and visualize coefficient estimates, risk ratios, and predicted probabilities.

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Log-binomial Regression

- **Outcome type:** binary, yes or no

- **Example outcomes:**
  - Food insecurity
  - Disease diagnosis for patient
  - Fracture

- **Population model**

$$\log(\mu) = \beta_0 + \beta_1 X$$

- **Interpretations**
  - We have log of probability on the left
  - So exponential of our coefficients will be **risk ratio**

Logistic regression vs. log-binomial regression

- Outcome is the same: still a binary variable
  - Still modeling with probability of event: $P(Y = 1|X)$
- Logistic = logit = log of odds
  - Aim to get **odds ratios** for interpretation (exponential of coefficients)
  - You *can* convert the odds ratios to risk ratios
- Log-binomial = log of probability
  - Aim to get **risk ratio** or prevalence ratio (exponential of coefficients)

STATS

EPI

$[0, \infty)$

systematic

- Disadvantage of log-binomial
  - Left hand side (aka $\log(\pi(X))$) is only a positive value, while right hand side can range from $-\infty$ to ∞
  - Prone to issues while trying to fit the model!

depending on your predictors

A few classes ago: GLOW Study with interactions

- **Outcome variable:** any fracture in the first year of follow up (FRACTURE: 0 or 1)
- **Risk factor/variable of interest:** history of prior fracture (PRIORFRAC: 0 or 1)
- **Potential confounder or effect modifier:** age (AGE, a continuous variable)

- Fit a logistic regression model with interactions:

$$\text{logit}\left(\pi(\mathbf{X})\right) = \beta_0 + \beta_1 \cdot I(\text{PF} = \text{``Yes''}) + \beta_2 \cdot Age + \beta_3 \cdot I(\text{PF} = \text{``Yes''}) \cdot Age$$

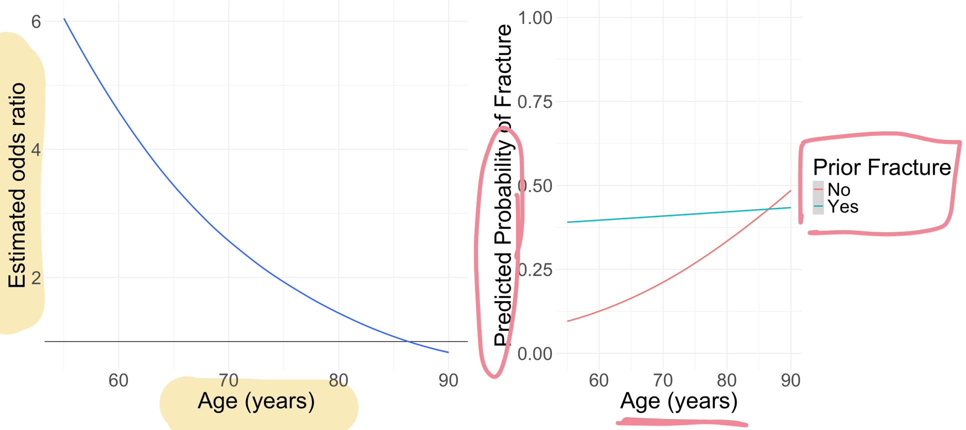
$$\text{logit}\left(\widehat{\pi}(\mathbf{X})\right) = \widehat{\beta}_0 + \widehat{\beta}_1 \cdot I(\text{PF} = \text{``Yes''}) + \widehat{\beta}_2 \cdot Age + \widehat{\beta}_3 \cdot I(\text{PF} = \text{``Yes''}) \cdot Age$$

$$\text{logit}\left(\widehat{\pi}(\mathbf{X})\right) = -1.376 + 1.002 \cdot I(\text{PF} = \text{``Yes''}) + 0.063 \cdot Age - 0.057 \cdot I(\text{PF} = \text{``Yes''}) \cdot Age$$

Logistic regression: Reporting results of GLOW Study with interactions

- Remember our main covariate is prior fracture, so we want to focuse on how age changes the relationship between prior fracture and a new fracture!

For individuals 69 years old, the estimated odds of a new fracture for individuals with prior fracture is 2.72 times the estimated odds of a new fracture for individuals with no prior fracture (95% CI: 1.70, 4.35). As seen in Figure 1 (a), the odds ratio of a new fracture when comparing prior fracture status decreases with age, indicating that the effect of prior fractures on new fractures decreases as individuals get older. In Figure 1 (b), it is evident that for both prior fracture statuses, the predict probability of a new fracture increases as age increases. However, the predicted probability of new fracture for those without a prior fracture increases at a higher rate than that of individuals with a prior fracture. Thus, the predicted probabilities of a new fracture converge at age [insert age here].



(a) Odds ratio of fracture outcome comparing prior fracture to no prior fracture

(b) Predicted probability of fracture

Figure 1: Plots of odds ratio and predicted probability from fitted interaction model

What would a log-binomial equivalent look like?

- For the same GLOW data, we could fit a log-binomial model:

link fn

$$\log\left(\widehat{\pi}(\mathbf{X})\right) = \widehat{\beta}_0 + \widehat{\beta}_1 \cdot I(\text{PF} = \text{“Yes”}) + \widehat{\beta}_2 \cdot Age + \widehat{\beta}_3 \cdot I(\text{PF} = \text{“Yes”}) \cdot Age$$

- What does this look like in the `glm` function?

```
glow_log_binom = glow %>% glm(formula = fracture ~ priorfrac + age_c + priorfrac*age_c,
                             family = binomial(link = "log"))
```

▶ Table for coefficient estimates

	term	estimate	std.error	statistic	p.value	conf.low	conf.high
$\hat{\beta}_0$	(Intercept)	−1.625	0.106	−15.269	0.000	−1.846	−1.427
$\hat{\beta}_1$	priorfracYes	0.727	0.159	4.570	0.000	0.404	1.034
$\hat{\beta}_2$	age_c	0.046	0.011	4.217	0.000	0.025	0.066
$\hat{\beta}_3$	priorfracYes:age_c	−0.043	0.016	−2.710	0.007	−0.074	−0.012

Let's start with a model with main effects only

- Just so we have a simpler model as we first demostrate the log-binomial:

$$\log\left(\widehat{\pi}(\mathbf{X})\right) = \widehat{\beta}_0 + \widehat{\beta}_1 \cdot I(\text{PF} = \text{“Yes”}) + \widehat{\beta}_2 \cdot Age$$

- Run a logistic regression model using `glm()`: *works!*

```
glow2_logistic = glow %>%
  glm(formula = fracture ~ priorfrac + age_c,
                              family = binomial)
```

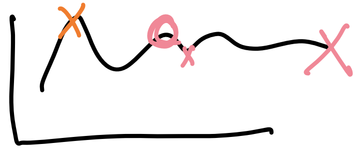
- Run a log-binomial model using `glm()`: *does not work!*

```
glow2_log_binom = glow %>%
  glm(formula = fracture ~ priorfrac + age_c,
                              family = binomial(link = "log"))
```

```
Error: no valid set of coefficients has been found: please supply starting values
```

How to fix this?

llk



$\log(\pi) \in [0, \infty)$

- We can fit the log-binomial model using `logbin()`
  - This function uses something called the EM algorithm to fit the data
  - Recall that `glm` uses an iteratively reweighted least squares algorithm

```
glow2_log_binom = logbin(formula = fracture ~ priorfrac + age_c,
              data = glow)
```

- Using `logbin` instead of `glm` will often fix the issue with fitting
  - But it does not work 100% of the time!

▶ Table for coefficient estimates from logbin()

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	−1.595	0.103	−15.428	0.000	−1.798	−1.393
priorfracYes	0.545	0.159	3.416	0.001	0.232	0.857
age_c	0.026	0.008	3.104	0.002	0.009	0.042

Model tests: same as logistic regression

- We would go through the same procedure as logistic regression
  - Because both models are rooted in the likelihood function!

- Use LRT to test multiple variables OR one categorical variable with multiple levels

  ↳ nested models

- Use Wald test to construct confidence intervals
  - Or test one continuous or one binary variable

Learning Objectives

1. Revisit the key distinctions between logistic and log-binomial regression.

2. Interpret and visualize coefficient estimates, risk ratios, and predicted probabilities.

Interpreting risk ratios from log-binomial regression

- Let's say we ran the following model:

$$\log\left(\widehat{\pi}(\mathbf{X})\right) = \widehat{\beta}_0 + \widehat{\beta}_1 \cdot I(\text{PF} = \text{“Yes”}) + \widehat{\beta}_2 \cdot Age$$

- $\exp(\widehat{\beta}_0)$: The estimated risk of fracture (outcome event) for someone who had no prior fracture and is the mean age of 69 years old

$\widehat{\pi}$ PF is $\exp(\widehat{\beta}_1)$ times $\widehat{\pi}$ no PF

- $\exp(\widehat{\beta}_1)$: The estimated risk of fracture for someone who had a prior fracture is $\exp(\widehat{\beta}_1)$ times the estimated risk of fracture for someone who did not have a prior fracture adjusting for age
- $\exp(\widehat{\beta}_2)$: For every one year increase in age, the estimated risk of fracture increases $\exp(\widehat{\beta}_2)$ times, adjusting for prior fracture.

*Recall that once I estimate the coefficients, my interpretations need to reflect that they are estimated by saying “estimated risk”

Interpretation of risk ratio: no interactions (1/2)

- If we have no interactions in the model (using `logbin`):

```
glow_main = logbin(formula = fracture ~ priorfrac + age_c, data = glow)
```

- Look at the exponential of the coefficients (risk ratio):

```
tidy(glow_main, conf.int = T, exponentiate = T) %>%
  gt() %>%
  tab_options(table.font.size = 35) %>%
  fmt_number(decimals = 3)
```

	term	estimate	std.error	statistic	p.value	conf.low	conf.high
$\exp(\hat\beta_0)$	(Intercept)	0.203	0.103	−15.428	0.000	0.166	0.248
$\exp(\hat\beta_1)$	priorfracYes	1.724	0.159	3.416	0.001	1.261	2.356
$\exp(\hat\beta_2)$	age_c	1.026	0.008	3.104	0.002	1.010	1.043

$(\hat\pi(PF))$ is 1.72 times $\hat\pi(\text{no } PF)$

estimated risk of fracture for someone who had prior fracture is 1.72 times that of someone w/out PF.

Interpretation of risk ratio: no interactions (2/2)

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	0.203	0.103	−15.428	0.000	0.166	0.248
priorfracYes	1.724	0.159	3.416	0.001	1.261	2.356
age_c	1.026	0.008	3.104	0.002	1.010	1.043

- $\exp(\widehat{\beta}_0)$: The estimated risk of fracture is 0.2 for someone who had no prior fracture and is the mean age of 69 years old
- $\exp(\widehat{\beta}_1)$: The estimated risk of fracture for someone who had a prior fracture is 1.72 times the estimated risk of fracture for someone who did not have a prior fracture, adjusting for age.
- $\exp(\widehat{\beta}_2)$: For every one year increase in age, the estimated risk of fracture is 1.03 times, adjusting for prior fracture.

  increases by 3%

  - OR: For every one year increase in age, the estimated risk of fracture increases 2.61%, adjusting for prior fracture.

Visualization: We can look at the log-risk

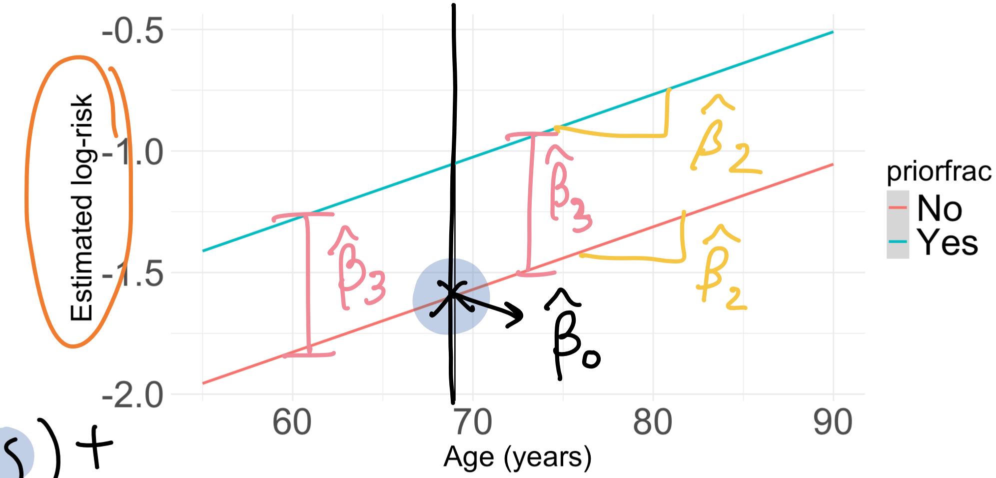
- This will help us understand the model that we fit
- Ultimately, we do not need a visualization of the risk ratio because the main effects model is constant across age

▶ Table of coefficient estimates

term	estimate	std.error	statistic	p.value
(Intercept)	−1.595	0.103	−15.428	0.000
priorfracYes	0.545	0.159	3.416	0.001
age_c	0.026	0.008	3.104	0.002

$$\log\left(\hat{\pi}(X)\right) = \hat{\beta}_0 + \hat{\beta}_1 I(PF=Yes) + \hat{\beta}_2 Age$$

▶ Code to make the following plot



Interactions: interpretations and visualizations

$$\log\left(\widehat{\pi}(\mathbf{X})\right) = \widehat{\beta}_0 + \widehat{\beta}_1 \cdot I(\text{PF} = \text{"Yes"}) + \widehat{\beta}_2 \cdot Age + \widehat{\beta}_3 \cdot I(\text{PF} = \text{"Yes"}) \cdot Age$$

```
glow3 = glow %>%
  mutate(interaction_PF_age = as.numeric(priorfrac)*age_c)

glow2_log_binom = logbin(formula = fracture ~ priorfrac + age_c + interaction_PF_age,
            data = glow3)

tidy(glow2_log_binom, conf.int = T) %>%
  gt() %>%
  tab_options(table.font.size = 30) %>%
  fmt_number(decimals = 3)
```

priorfrac * age_c

	term	estimate	std.error	statistic	p.value	conf.low	conf.high
$\hat{\beta}_0$	(Intercept)	−1.596	0.104	−15.414	0.000	−1.799	−1.393
$\hat{\beta}_1$	priorfracYes	0.576	0.170	3.382	0.001	0.242	0.910
$\hat{\beta}_2$	age_c	0.037	0.026	1.440	0.150	−0.013	0.088
$\hat{\beta}_3$	interaction_PF_age	−0.009	0.017	−0.546	0.585	−0.042	0.023

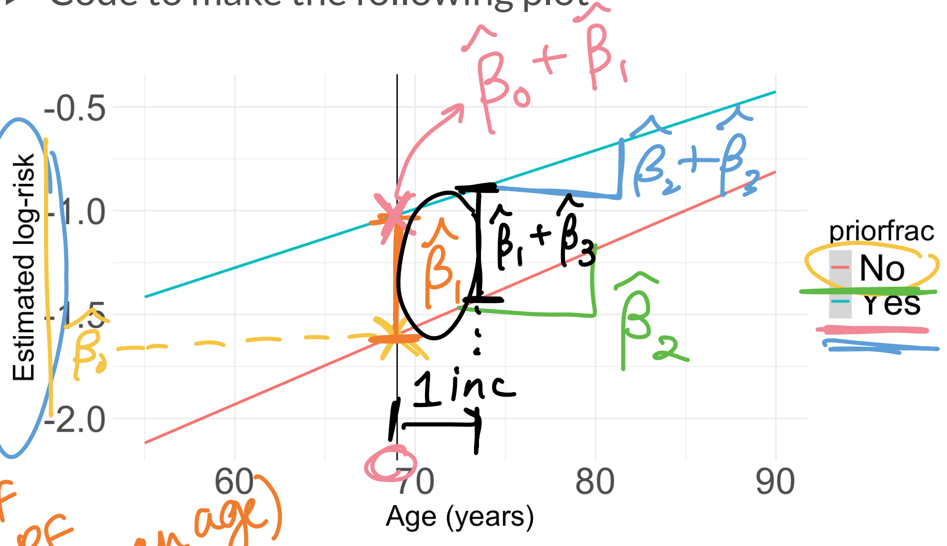
Visualization of interaction: We can look at the log-risk

- This will help us understand the model that we fit
- Ultimately, we do not need a visualization of the risk ratio because the main effects model is constant across age

▶ Table of coefficient estimates

term	estimate	std.error	statistic	p.value
(Intercept)	−1.596	0.104	−15.414	0.000
priorfracYes	0.576	0.170	3.382	0.001
age_c	0.037	0.026	1.440	0.150
interaction_PF_age	−0.009	0.017	−0.546	0.585

▶ Code to make the following plot



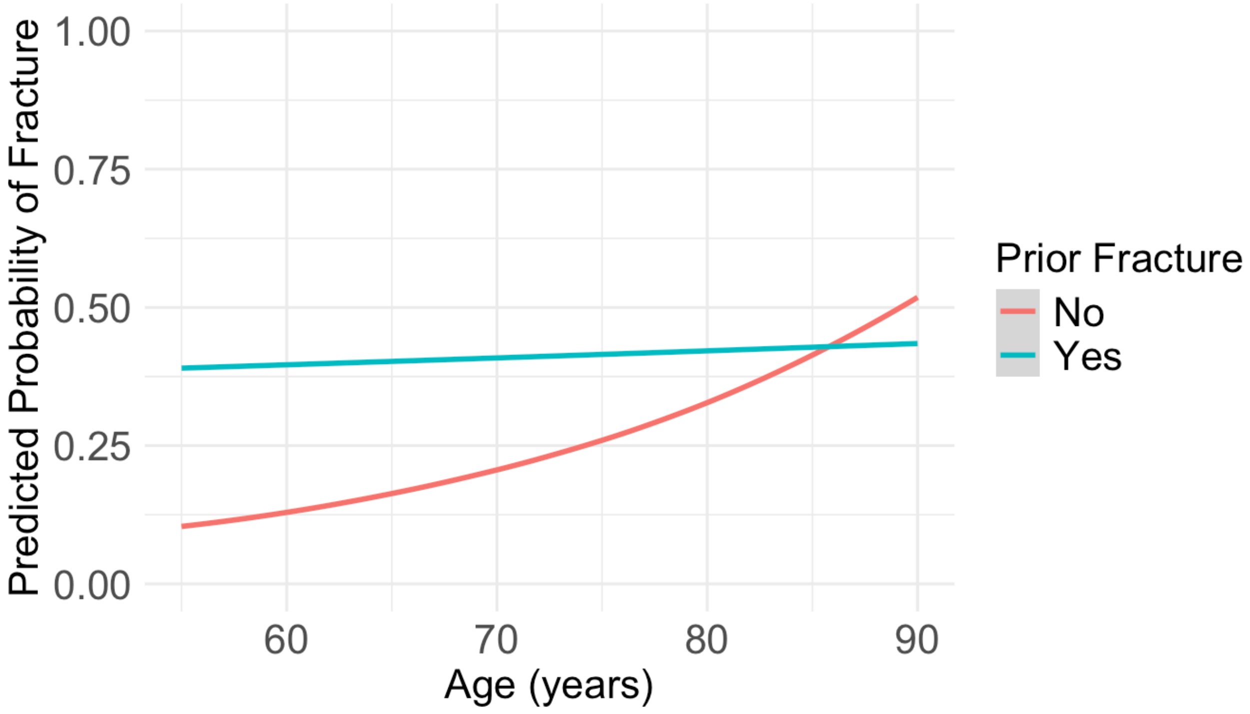
diff b/w log-risk comparing PF to No PF (@ mean age)

$$\log\left(\widehat{\pi}(\mathbf{X})\right) = \widehat{\beta}_0 + \widehat{\beta}_1 \cdot \underbrace{I(\text{PF} = \text{“Yes”})}_{=0} + \widehat{\beta}_2 \cdot Age + \widehat{\beta}_3 \cdot \underbrace{I(\text{PF} = \text{“Yes”}) \cdot Age}_{=0}$$

Predicted probabilities and visualizations

- Does this look familiar?? It should!
  - The predicted probabilities from the logistic regression and the log-binomial regression should be the same!

▶ Code to make the following plot



Final words on log-binomial vs logistic

- Log-binomial is more common in field of Epidemiology

- I prefer to use logistic regression because it is more stable AND I can calculate risk ratio from my odds ratio and predicted probability

- If you use $OR \approx RR$, then logistic regression will over-estimate the RR
  - BUT you can still use logistic regression to calculate the RR
  - Find the predicted probabilities for two groups you are comparing and then calculate the RR

More resources

- Log-binomial models: exploring failed convergence
- Parameter estimation and goodness-of-fit in log binomial regression
    - Needs PSU login for full access