

# Lesson 8: Tests for GLMs using Likelihood function

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2025-04-21

## I've never explicitly said this...

- Because we are in a public health class, we are often analyzing data with sensitive outcomes
- If you ever need a moment in class because of our topic, feel free to just step out or leave and privately view the lecture
- If you need extra time on your assignments because you have an emotional response to lectures/homework/lab, just let me know! Extenuating circumstance!

# Learning Objectives

1. Use the Wald test to test the significance of an estimated coefficient through confidence intervals.
2. Articulate how the Wald test, Score test, and likelihood ratio test (LRT) calculates a test statistic using the likelihood function.
3. Use the Likelihood ratio test to test the significance of estimated coefficients through formal hypothesis testing.
4. Understand when and how to use each test: Wald, Score, and Likelihood ratio

# Connection between tests in linear models and GLMs

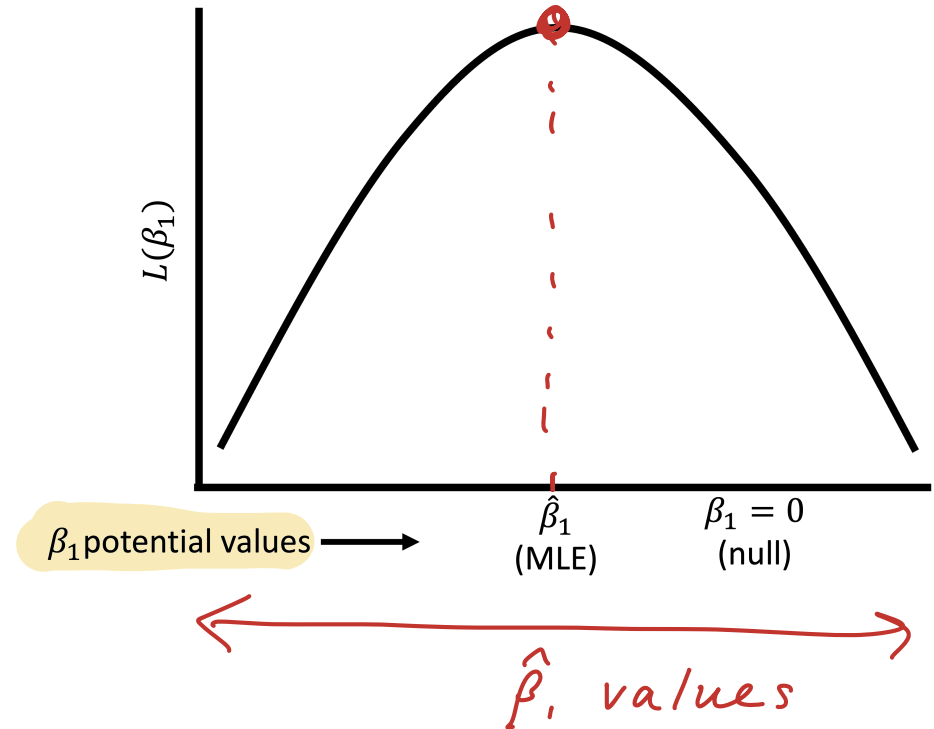
- In linear regression, we used **ordinary least squares (OLS)** to find the best fit model, so we could use the following tests:
  - **t-test for single coefficients**
  - **F-test for single coefficients or groups of coefficients**
- These tests **hinge on the Mean Squared Error (MSE)** which we **minimized in OLS** and the LINE assumptions
  
- In GLMs, when we use **maximum likelihood estimation (MLE)**, we **cannot** use t-tests or F-tests
  - Because we are now using **likelihood** to find our estimates (not OLS)
- But we have **parallel tests in MLE!!**
  - **t-test → Wald test**
  - **F-test → Likelihood ratio test**

# Revisit the likelihood function

- **Likelihood function:** expresses the probability of the observed data as a function of the unknown parameters
  - Function that enumerates the likelihood (similar to probability) that we observe the data **across the range of potential values of our coefficients**
- We often compare likelihoods to see what estimates are more likely given our data
- Plot to right is a simplistic view of likelihood
  - I have flattened the likelihood that would be a function of  $\beta_0$  and  $\beta_1$  into a 2D plot (instead of 3D:  $\beta_0$  vs.  $\beta_1$  vs.  $L(\beta_0, \beta_1)$ )
- I use  $L$  to represent the log-likelihood and  $l$  to represent the likelihood

nat.  
log

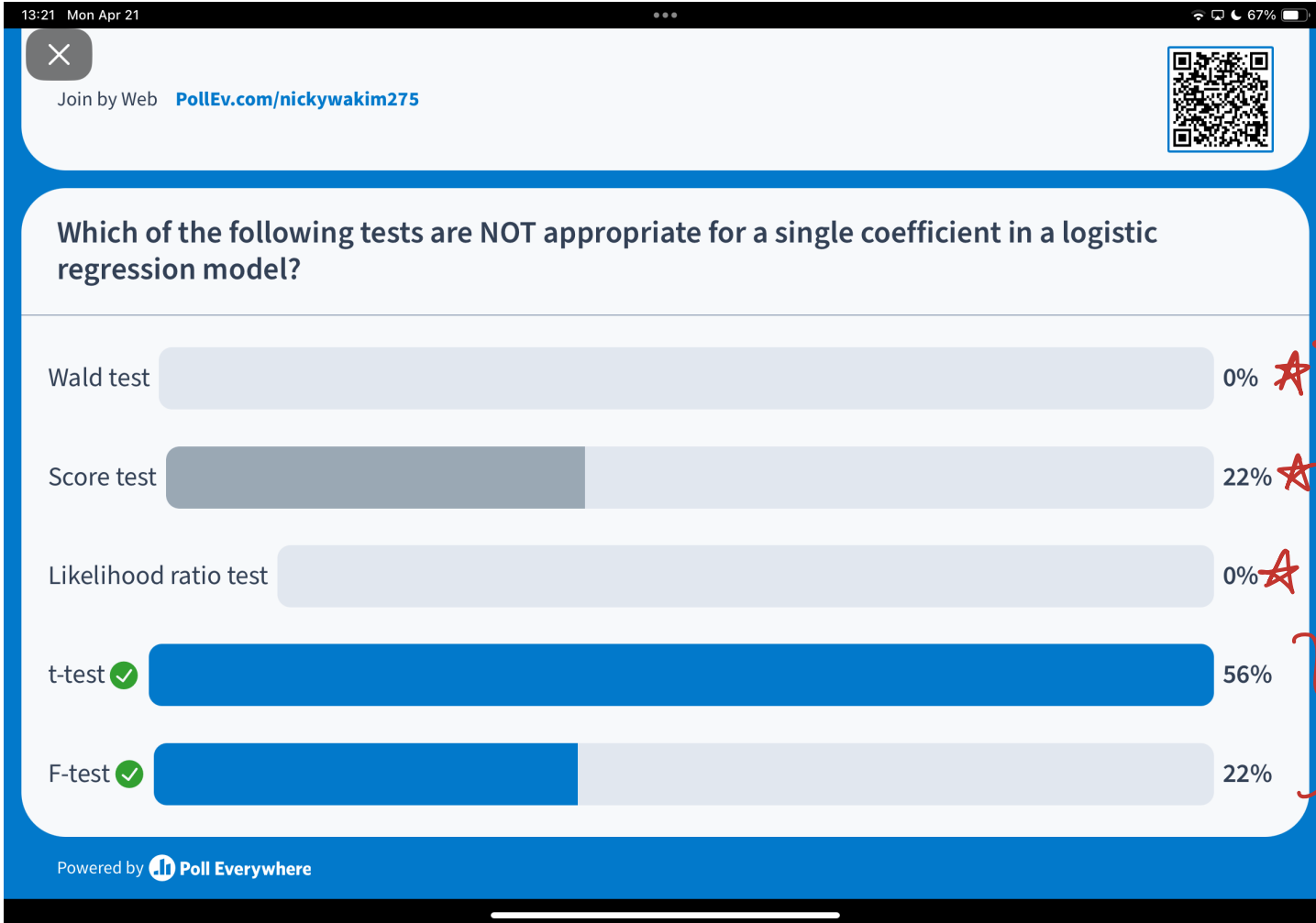
$$L = \log(l)$$
$$L = \log(1)$$



# Introduction to three tests in GLM

- To introduce these three tests, we will work on a **single coefficient**
  - To be clear: the *Likelihood ratio test can be extended to more coefficients*
- Let's say we fit a GLM using MLE
  - We will continue to use **logistic regression as our working example**
  
- Now we want to run a hypothesis test for an **individual coefficient  $j$** :
  - $H_0 : \beta_j = 0$
  - $H_1 : \beta_j \neq 0$
- Three potential tests that we use with a Likelihood function are:
  - **Wald test**
  - **Score test**
  - **Likelihood ratio test (LRT)**

# Poll Everywhere Question 1



used w/ MLE

used w/ OLS  
(cannot be used  
w/ MLE)

# Revisit previous model with late stage BC diagnosis and age

- Simple logistic regression model:

$$\text{logit}(\pi(\text{Age})) = \beta_0 + \beta_1 \cdot \text{Age}$$

Don't forget:  $\pi(\text{Age}) = P(Y = 1 | \text{Age})$

```
1 bc_reg = bc %>% glm(formula = Late_stage_diag ~ Age_c, family = binomial)
2 tidy(bc_reg, conf.int=T) %>% gt() %>% tab_options(table.font.size = 38) %>%
3   fmt_number(decimals = 3)
```

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	-0.989	0.023	-42.637	0.000	-1.035	0.944
Age_c	0.057	0.003	17.780	0.000	0.051	0.063

$\hat{\beta}_0$   
 $\hat{\beta}_1$

Coefficient estimates

what test is this?  
Wald

$$\text{logit}(\hat{\pi}(\text{Age})) = -0.989 + 0.057 \text{Age}$$
$$\hat{\beta}_0 + \hat{\beta}_1 \text{Age}$$

# Learning Objectives

1. Use the Wald test to test the significance of an estimated coefficient through confidence intervals.
2. Articulate how the **Wald test**, Score test, and likelihood ratio test (LRT) calculates a test statistic using the likelihood function.
3. Use the Likelihood ratio test to test the significance of estimated coefficients through formal hypothesis testing.
4. Understand when and how to use each test: Wald, Score, and Likelihood ratio

## Wald test (1/3)

- Very similar to a t-test!
  - But slightly different because it based in our likelihood function
- Assumes test statistic  $W$  follows a standard normal distribution under the null hypothesis
- Test statistic:

$$W = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)} \sim N(0, 1)$$

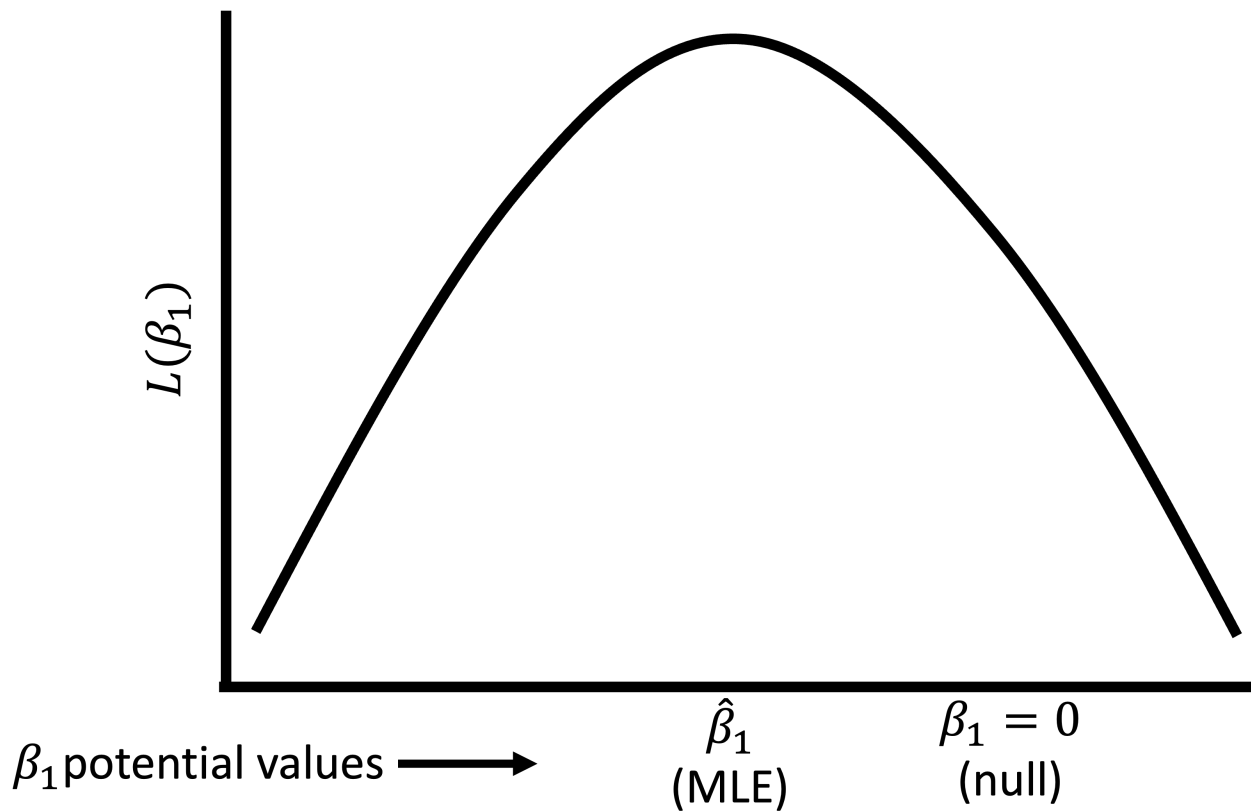
- where  $\hat{\beta}_j$  is a MLE of coefficient  $j$
- 95% Wald confidence interval:

$$\hat{\beta}_1 \pm 1.96 \cdot SE_{\hat{\beta}_j}$$

- The Wald test is a routine output in R (`summary()` of `glm()` output)
  - Includes  $SE_{\hat{\beta}_j}$  and can easily find confidence interval with `tidy()`

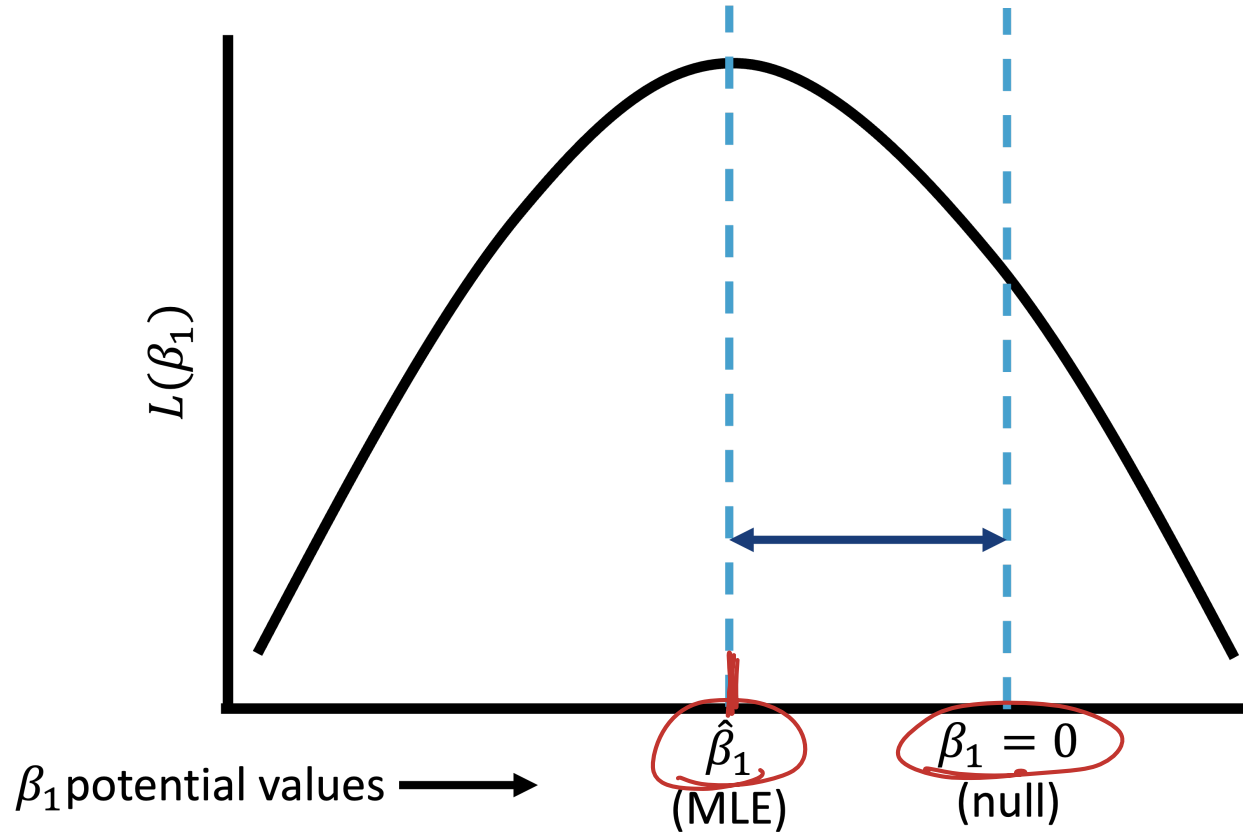
## Wald test (2/3)

- Wald test measures the “difference” between  $\beta_1 = 0$  and  $\hat{\beta}_1$  on the x-axis



## Wald test (3/3)

- Wald test measures the “difference” between  $\beta_1 = 0$  and  $\hat{\beta}_1$  on the x-axis



# Wald test procedure with confidence intervals

1. Set the **level of significance**  $\alpha$
2. Specify the **null** ( $H_0$ ) and **alternative** ( $H_A$ ) **hypotheses**
  - In symbols
  - In words
  - Alternative: one- or two-sided?
3. Calculate the **confidence interval** and **determine if it overlaps with null**
  - Overlap with null (usually 0 for coefficient) = fail to reject null
  - No overlap with null (usually 0 for coefficient) = reject null
4. Write a **conclusion** to the hypothesis test
  - What is the estimate and its confidence interval?
  - Do we reject or fail to reject  $H_0$ ?
  - Write a conclusion in the context of the problem

# Example: BC diagnosis and age

## Wald test for age coefficient

Interpret the coefficient for age in our model of late stage breast cancer diagnosis.

### Needed steps:

1. Set the **level of significance**  $\alpha$
2. Specify the **null** ( $H_0$ ) and **alternative** ( $H_A$ ) **hypotheses**
3. Calculate the **confidence interval** and **determine if it overlaps with null**
4. Write a **conclusion** to the hypothesis test

### Note

I don't want us to get fixated on this interpretation. This is more to introduce the process, BUT it's MUCH better to interpret the coefficient in terms of OR (next class).

# Example: BC diagnosis and age

## Wald test for age coefficient

Interpret the coefficient for age in our model of late stage breast cancer diagnosis.

1. Set the **level of significance**  $\alpha$

$$\alpha = 0.05$$

2. Specify the **null** ( $H_0$ ) and **alternative** ( $H_A$ ) hypotheses

$$H_0 : \beta_{Age} = 0$$

$$H_1 : \beta_{Age} \neq 0$$

# Example: BC diagnosis and age

## Wald test for age coefficient

Interpret the coefficient for age in our model of late stage breast cancer diagnosis.

### 3. Calculate the confidence interval and determine if it overlaps with null

```
1 library(epiDisplay)
2 bc_reg = glm(Late_stage_diag ~ Age_c, data = bc, family = binomial)
3 tidy(bc_reg, conf.int=T) %>% gt() %>% tab_options(table.font.size = 35) %>%
4   fmt_number(decimals = 3)
```

term	estimate	std.error	statistic	p.value	conf.low	conf.high
(Intercept)	-0.989	0.023	-42.637	0.000	-1.035	-0.944
Age_c	0.057	0.003	17.780	0.000	0.051	0.063

$\hat{\beta}_i$

does NOT  
overlap 0

$\exp(\hat{\beta}_i)$   $\rightarrow \exp(0) = 1$   $\rightarrow$  check overlap w/ 1

# Example: BC diagnosis and age

## Wald test for age coefficient

Interpret the coefficient for age in our model of late stage breast cancer diagnosis.

### 4. Write a **conclusion** to the hypothesis test

For every one year increase in age, the log-odds of late stage breast cancer diagnosis increases 0.057 (95% CI: 0.051, 0.063).

There is sufficient evidence that age and breast cancer diagnosis are associated.

For every one yr inc in age,  
the odds of LSBC diagnosis is 1.058 times  
(95% CI  $\downarrow$  ,  $\downarrow$ )  
 $\exp(0.051)$ ,  $\exp(0.063)$


### Note

I don't want us to get fixated on this interpretation. This is more to introduce the process, BUT it's MUCH better to interpret the coefficient in terms of OR (next class).

# Poll Everywhere Question 2

13:34 Mon Apr 21

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**For a multi-level categorical variable like race/ethnicity, can we use the Wald test to determine if breast cancer diagnosis and race/ethnicity are associated?**

~~Yes, we can look at the significance of each coefficient~~ 25%

~~Yes, if one coefficient is significant that we can conclude there is association~~ 13%

**No, we need to use a test for the addition of the entire variable (like LRT)** 63%

No, it's too hard to look at each coefficient 0%

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model w/  
R/E

model w/out  
R/E

# Learning Objectives

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2. Articulate how the Wald test, **Score test**, and likelihood ratio test (LRT) calculates a test statistic using the likelihood function.
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# Score test

- Score test **does not require the computation of MLE** for  $\beta_1$ , while both likelihood test and Wald test do
  - Only need to know  $\beta_1$  under the null

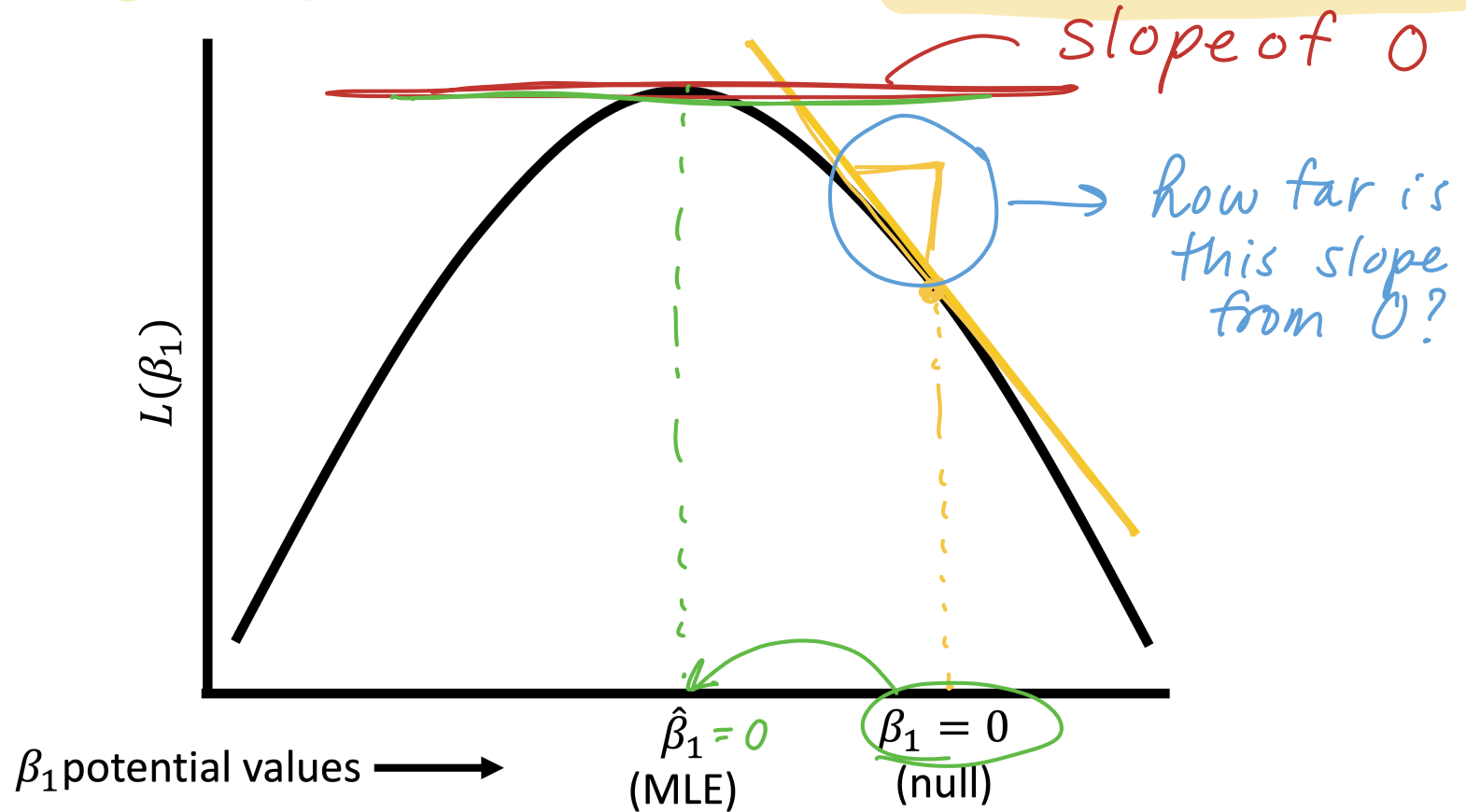
- Score test is based on the first and second derivatives of the log-likelihood under the null hypothesis:

$$S = \frac{\sum_{i=1}^n x_i (y_i - \bar{y})}{\sqrt{\bar{y}(1 - \bar{y}) \sum_{i=1}^n (x_i - \bar{x})^2}} \sim N(0, 1)$$

- Translates to the **slope of the tangent line** at  $\beta_1 = 0$
- In my experience, this is the test I've seen the least: won't show an example with it

# Score test

- Score test measures the “difference” between  $\beta_1 = 0$  and  $\hat{\beta}_1$  by measuring slope of the tangent line at  $\beta_1 = 0$



# Learning Objectives

1. Use the Wald test to test the significance of an estimated coefficient through confidence intervals.

2. Articulate how the Wald test, Score test, and **likelihood ratio test (LRT)** calculates a test statistic using the likelihood function.

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# Likelihood ratio test (1/3)

- Likelihood ratio test (LRT) answers the question:
  - For a specific covariate, **which model tell us more about the outcome variable:** the model including the covariate or the model omitting the covariate?
  - Aka: Which model is more likely given our data: model including the covariate or the model omitting the covariate?
- **Very similar to the F-test:** test significance by comparing different, nested models
- Important: LRT can be used conduct hypothesis tests for a **single coefficient** or **multiple coefficients**
  - Just like F-test, we can test a single coefficient, continuous/binary covariate, multi-level covariate, or multiple covariates

*single variable*

## Likelihood ratio test (2/3)

- To assess the significance of a continuous/binary covariate's coefficient in the simple logistic regression, we compare the **deviance (D) with and without the covariate**

$$G = D(\text{model without } x) - D(\text{model with } x)$$

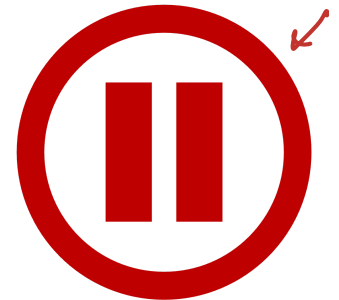
*next few slides* [ ]

*Deviance* ↑

- For a continuous or binary variable, this is equivalent to test:  $H_0 : \beta_j = 0$  vs.  $H_1 : \beta_j \neq 0$
- Test statistic for LRT:

$$G = -2 \ln \left[ \frac{\text{likelihood without } x}{\text{likelihood with } x} \right] = 2 \ln \left[ \frac{l(\hat{\beta}_0, \hat{\beta}_1)}{l(\hat{\beta}_0)} \right]$$

*multi-level:  
null is  
that all  
 $\beta_s = 0$*



# LRT: what is Deviance?

• **Deviance** quantifies the difference in likelihoods between a **fitted** and **saturated** model

▪ **Fitted model:**

- Your proposed fitted model

▪ **Saturated model:**

- A model that contains as many parameters as there are data points = **perfect fit**
- Basically every individual has their own covariate
- Perfect fit = maximum possible likelihood

→ 512/612: model selection generalizability

• All fitted models will have likelihood less than saturated model



# LRT: what is Deviance?

- The **deviance** is mathematically defined as:

$$D = -2[L_{\text{fitted}} - L_{\text{saturated}}]$$

- An alternative way to write it is:

$$D = -2 \ln \left[ \frac{\text{likelihood of the fitted model}}{\text{likelihood of the saturated model}} \right]$$

- Using '-2' is to make the deviance follow a chi-square distribution

$$\log(a) - \log(b)$$
$$\log(l_{\text{fitted}}) - \log(l_{\text{sat}})$$
$$\log\left(\frac{a}{b}\right)$$



# Deviance to Likelihood Ratio Test

- In the LRT, we are NOT comparing the likelihood of saturated model to the fitted model
- We ARE comparing the Deviance of the model with  $X$  and the model without  $X$ 
  - We just use the saturated model to calculate Deviance
  - Both are considered fitted models with their own respective Deviance

- So our LRT statistic is:

$$G = D(\text{model without } x) - D(\text{model with } x)$$

- But deviance is calculated as:

$$\text{w/out } X \quad D = -2[L_{\text{model w/out } X} - L_{\text{saturated}}]$$

$$D_{\text{model w/ } X} = -2[L_{\text{model w/ } X} - L_{\text{sat}}]$$



## For reference

$$G = D(\text{model without } x) - D(\text{model with } x)$$

$$G = -2\ln \left[ \frac{\text{likelihood of model without } x}{\text{likelihood of saturated model}} \right] - \left( -2\ln \left[ \frac{\text{likelihood of model with } x}{\text{likelihood of saturated model}} \right] \right)$$

$$G = -2\ln \left[ \frac{\text{likelihood of model without } x}{\text{likelihood of saturated model}} \right] \times \frac{\text{likelihood of saturated model}}{\text{likelihood of model with } x}$$

$$G = -2\ln \left[ \frac{\text{likelihood of model without } x}{\text{likelihood of model with } x} \right]$$

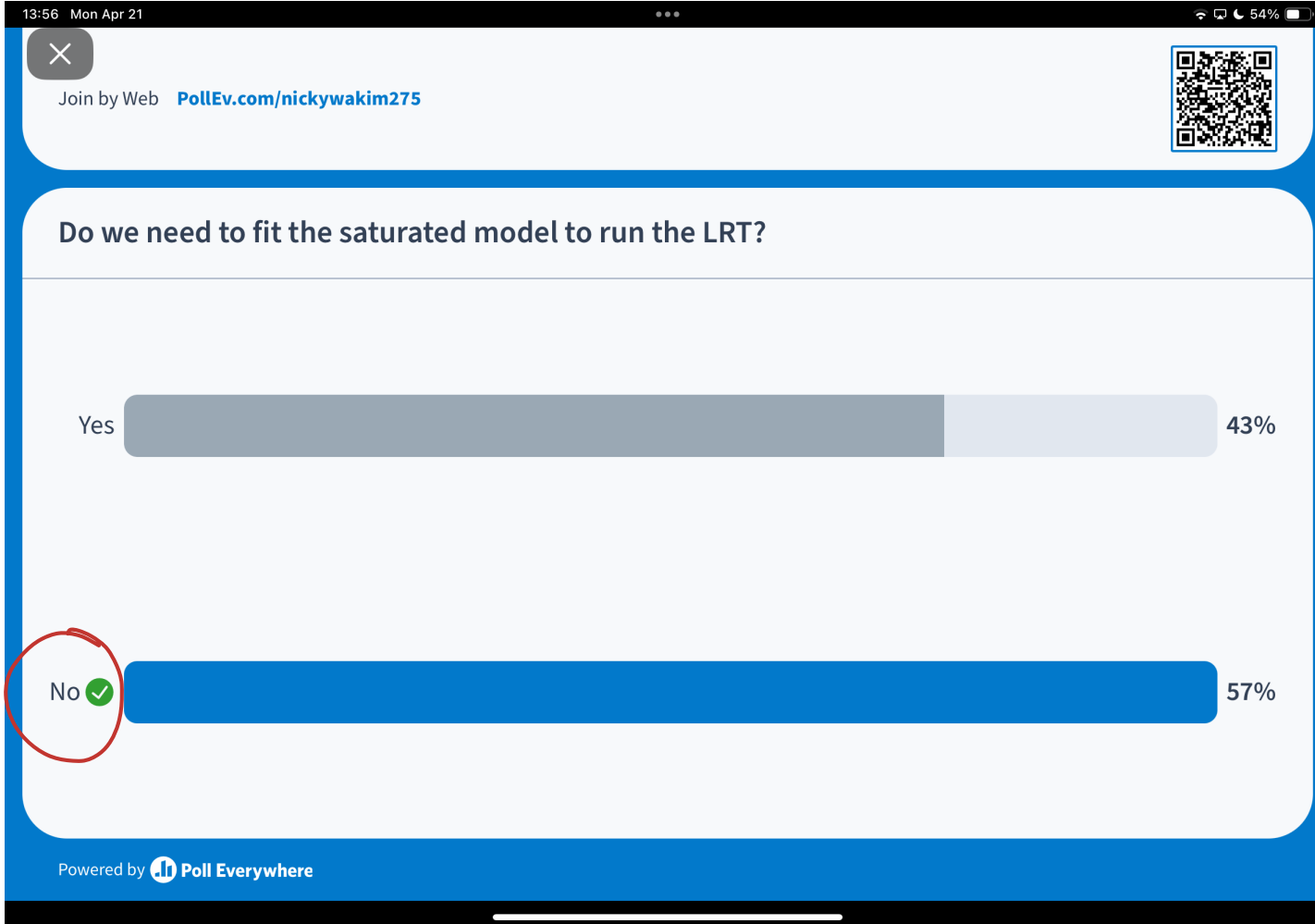
$$G = 2\ln \left[ \frac{l(\hat{\beta}_0, \hat{\beta}_1)}{l(\hat{\beta}_0)} \right] \begin{array}{l} \rightarrow \text{w/ } x \\ \rightarrow \text{w/out } x \end{array}$$

$$\begin{aligned} & -\log\left(\frac{a}{b}\right) \\ & = \log\left(\frac{b}{a}\right) \end{aligned}$$

$$\begin{aligned} & \log(a/c) - \\ & \log(b/d) \\ & = \log\left(\frac{a \cdot d}{b \cdot c}\right) \end{aligned}$$



# Poll Everywhere Question 3



$l(\hat{\beta}_1, \hat{\beta}_0) \checkmark$   
 $l(\hat{\beta}_0) \checkmark$   
can get this  
in  $\mathcal{R}$

## Likelihood ratio test (2/3)

- To assess the significance of a continuous/binary covariate's coefficient in the simple logistic regression, we compare the **deviance (D) with and without the covariate**

$$G = D(\text{model without } x) - D(\text{model with } x)$$

- For a continuous or binary variable, this is equivalent to test:  $H_0 : \beta_j = 0$  vs.  $H_1 : \beta_j \neq 0$
- Test statistic for LRT:

$$G = -2\ln \left[ \frac{\text{likelihood without } x}{\text{likelihood with } x} \right] = 2\ln \left[ \frac{l(\hat{\beta}_0, \hat{\beta}_1)}{l(\hat{\beta}_0)} \right]$$



## Likelihood ratio test (3/3)

- Under the null hypothesis, with adequate sample size, LRT statistic follows a chi-square distribution:

$$\rightarrow G \sim \chi^2(df)$$

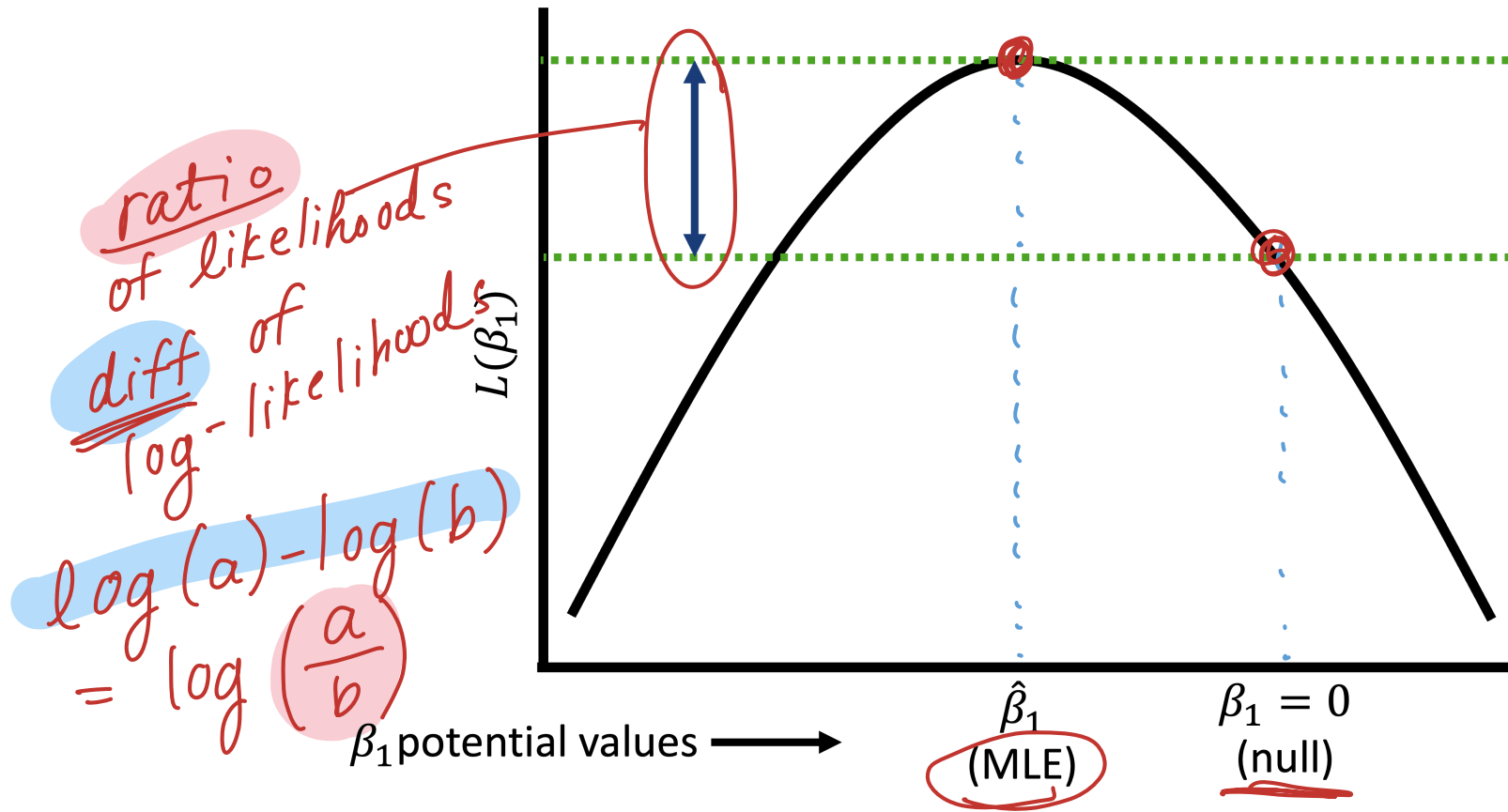
- $df = (\underbrace{\# \text{coefficients in larger model}}_{w/\chi}) - (\underbrace{\# \text{coefficients in smaller model}}_{w/out \chi})$

- If we are testing a single coefficient, like age, then  $df = 1$

- We can use the `pchisq()` if you calculate  $G$  first, but there's an R function to run an LRT

# Likelihood ratio test

- Likelihood ratio test measures the <sup>r</sup>difference <sup>r</sup>of likelihoods (y-axis) between  $\beta_1 = 0$  and  $\hat{\beta}_1$



# LRT procedure

1. Set the **level of significance**  $\alpha$
2. Specify the **null** ( $H_0$ ) and **alternative** ( $H_A$ ) **hypotheses**
  - In symbols
  - In words
  - Alternative: one- or two-sided?
3. Calculate the **test statistic** and **p-value**
4. Write a **conclusion** to the hypothesis test
  - Do we reject or fail to reject  $H_0$ ?
  - Write a conclusion in the context of the problem

# Example: BC diagnosis and age

## LRT

Determine if the model including age is more likely than model without age. Aka: Is age associated with late stage breast cancer diagnosis?

### Needed steps:

1. Set the **level of significance**  $\alpha$
2. Specify the **null** ( $H_0$ ) and **alternative** ( $H_A$ ) hypotheses
3. Calculate the **test statistic** and **p-value**
4. Write a **conclusion** to the hypothesis test

# Example: BC diagnosis and age

## LRT

Determine if the model including age is more likely than model without age. Aka: Is age associated with late stage breast cancer diagnosis?

1. Set the **level of significance**  $\alpha$

$$\alpha = 0.05$$

2. Specify the **null** ( $H_0$ ) and **alternative** ( $H_A$ ) hypotheses

$$\begin{array}{l} H_0 : \beta_{Age} = 0 \\ H_1 : \beta_{Age} \neq 0 \end{array} \left. \vphantom{\begin{array}{l} H_0 \\ H_1 \end{array}} \right\} \begin{array}{l} \text{model w/out age} \\ \text{model w/ age} \end{array}$$

# Example: BC diagnosis and age

## LRT

Determine if the model including age is more likely than model without age. Aka: Is age associated with late stage breast cancer diagnosis?

### 3. Calculate the **test statistic** and **p-value**

```
1 library(lmtest)
2 bc_age = glm(Late_stage_diag ~ Age_c, data = bc, family = binomial)
3 bc_int = glm(Late_stage_diag ~ 1, data = bc, family = binomial)
4 lmtest::lrtest(bc_age, bc_int)
```

Likelihood ratio test

Model 1: Late\_stage\_diag ~ Age\_c

Model 2: Late\_stage\_diag ~ 1

	#Df	LogLik	Df	Chisq	Pr(>Chisq)
1	2	-5754.8			
2	1	-5930.5	-1	351.27	< 2.2e-16 ***

---

Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

## Example: BC diagnosis and age

LRT

Determine if the model including age is more likely than model without age. Aka: Is age associated with late stage breast cancer diagnosis?

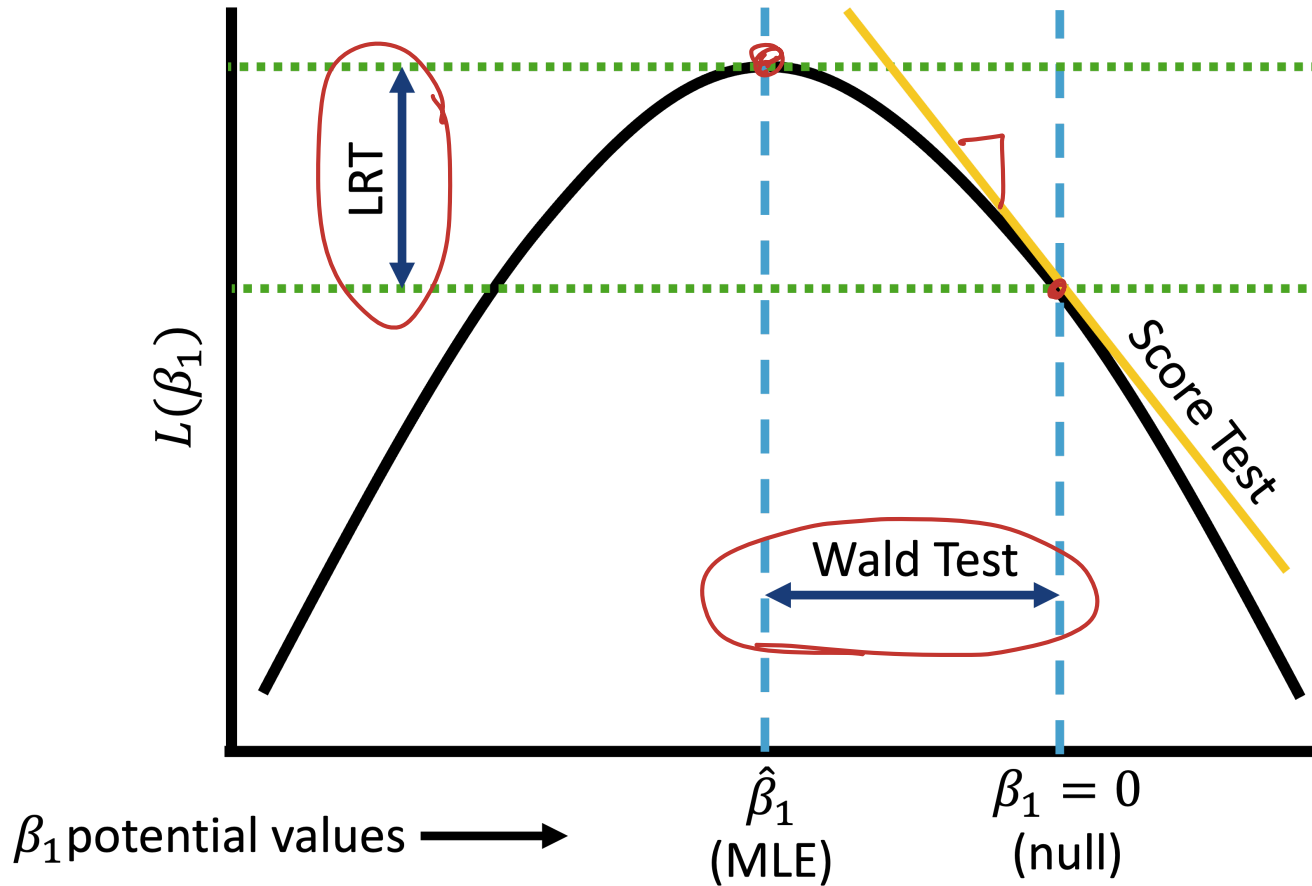
4. Write a **conclusion** to the hypothesis test

We reject the null hypothesis that the coefficient corresponding to age is 0 ( $p - value \ll 0.05$ ). There is sufficient evidence that there is an association between age and late stage breast cancer diagnosis.

# Learning Objectives

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2. Articulate how the Wald test, Score test, and likelihood ratio test (LRT) calculates a test statistic using the likelihood function.
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4. Understand when and how to use each test: Wald, Score, and Likelihood ratio

# All three tests together



# Which test to use?

- All three tests are asymptotically equivalent
  - As sample approaches infinity
- For testing significance of single covariate coefficient:
  - **LRT**
    - Wald and score are only approximations of LRT
    - For smaller samples, LRT better
  - **Wald test** is very convenient
    - Automatically performed in R
    - Does not need to estimate two models (LRT does)
    - Good for constructing confidence intervals of coefficients and odds ratios
  - **Score test**
    - Does not need to estimate two models (LRT does), but does not incorporate info from MLE
    - I don't really see people use this...

models w/ & w/ mult covariates  
(or single covariate multi-level)

one covariate (cont or binary)

# Poll Everywhere Question 4

14:25 Mon Apr 21

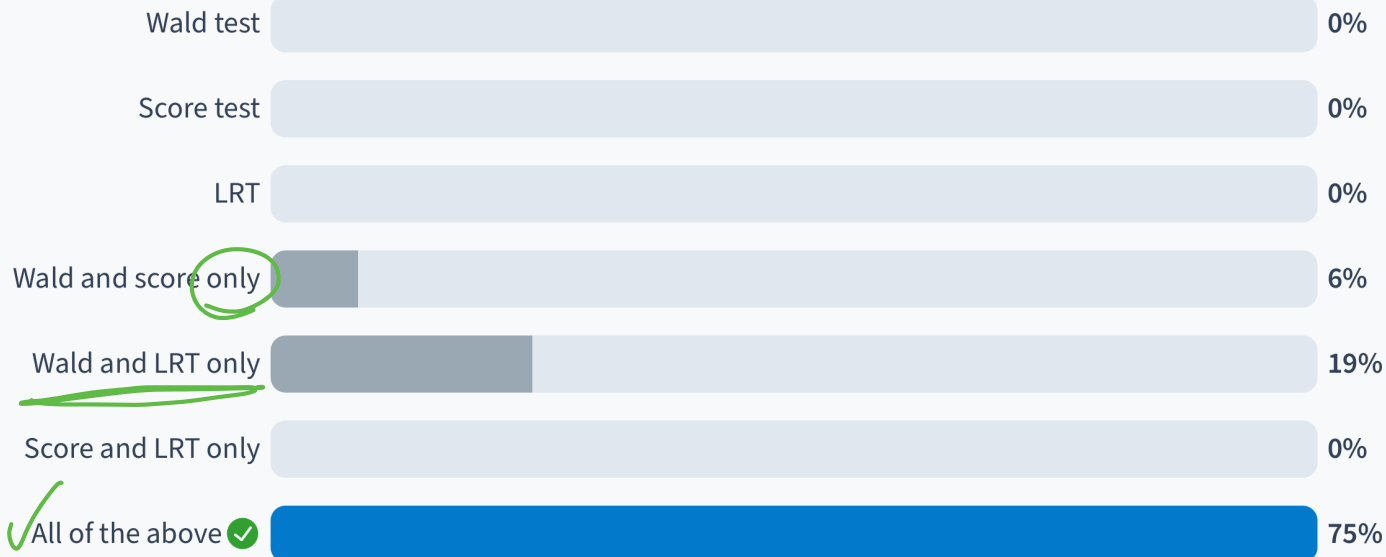
44%



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Which of the following test(s) can be used to test significance of single coefficient?



# Poll Everywhere Question 5

14:28 Mon Apr 21

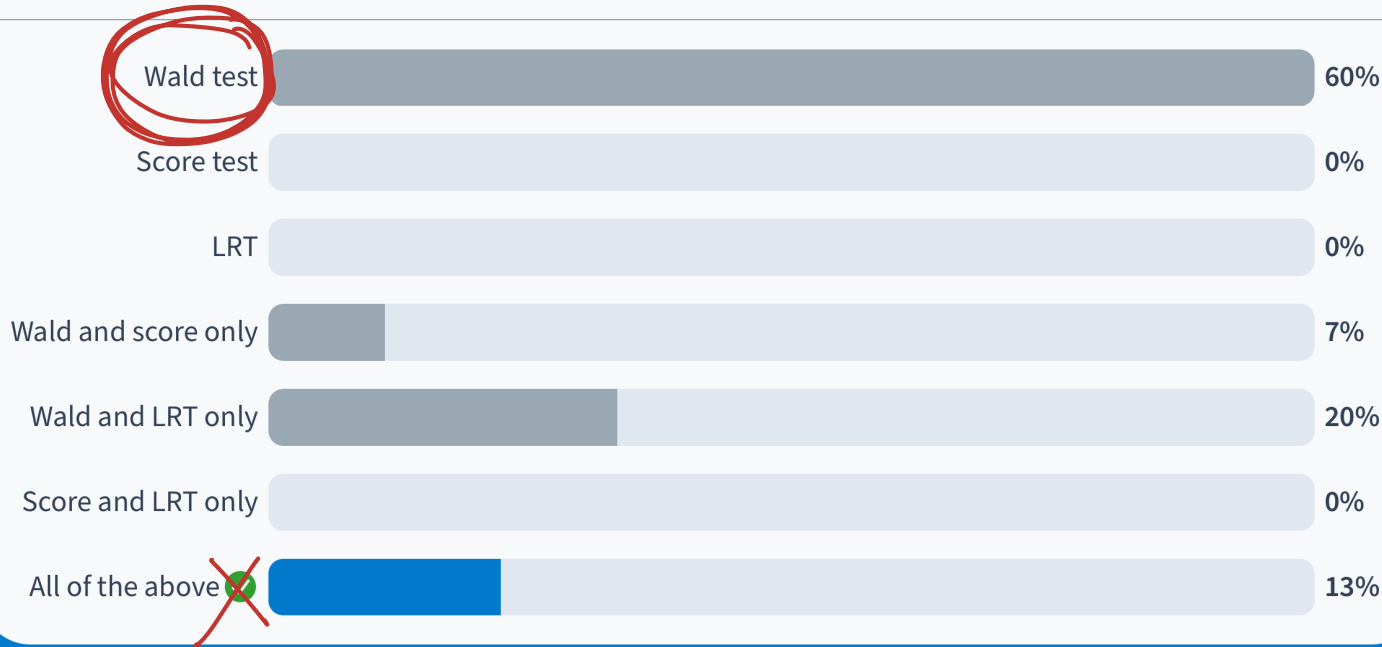
43%



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Which test(s) can be used to report confidence interval for a single coefficient?



*Really annoying for score & LRT*

# Poll Everywhere Question 6

14:30 Mon Apr 21

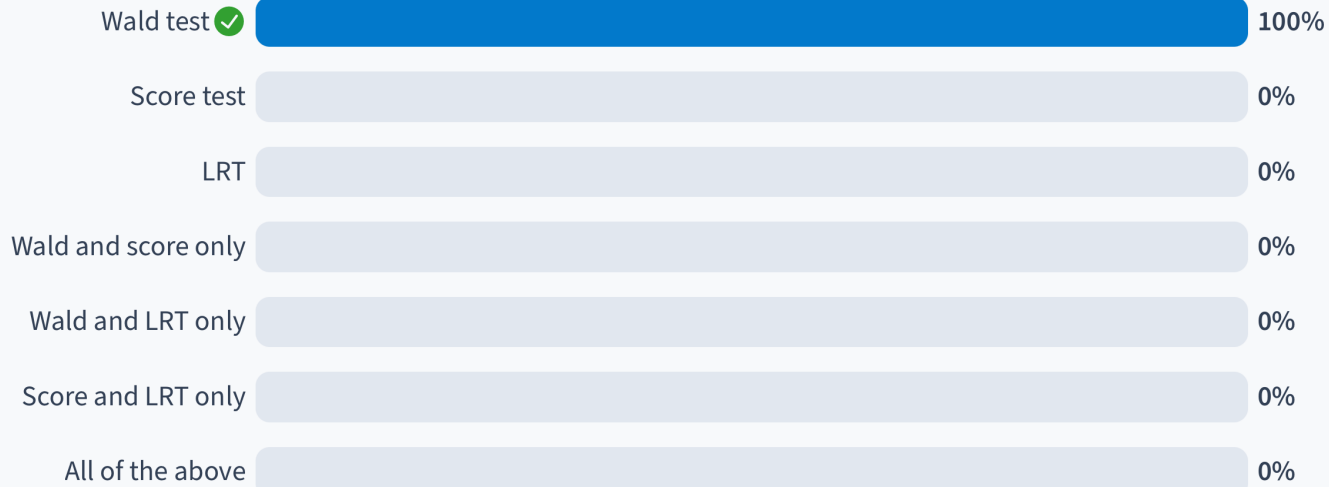
42%



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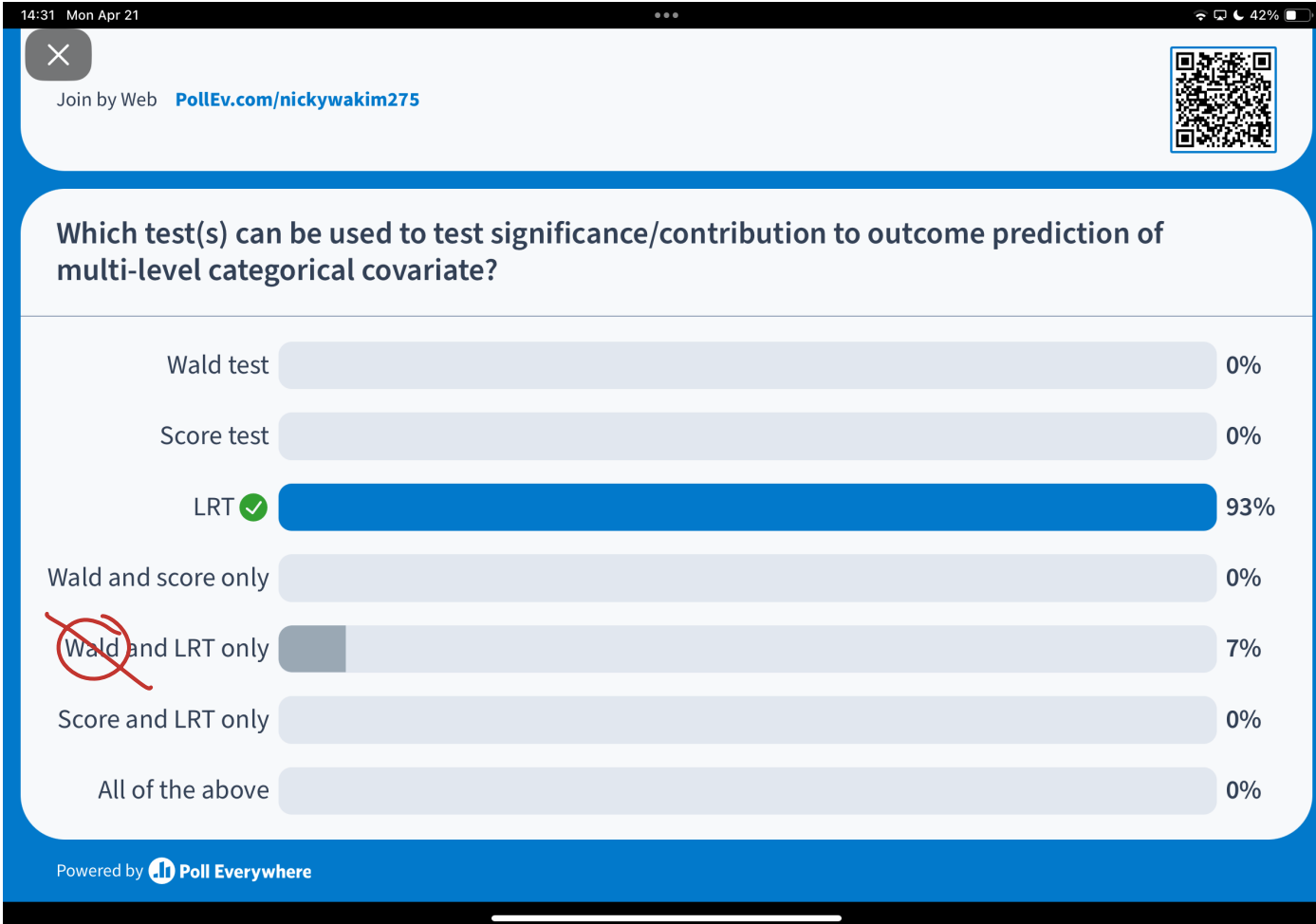


Which test(s) are used in the confidence interval automatically reported by R for a single coefficient?



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# Poll Everywhere Question 7



# Poll Everywhere Question 8

14:33 Mon Apr 21

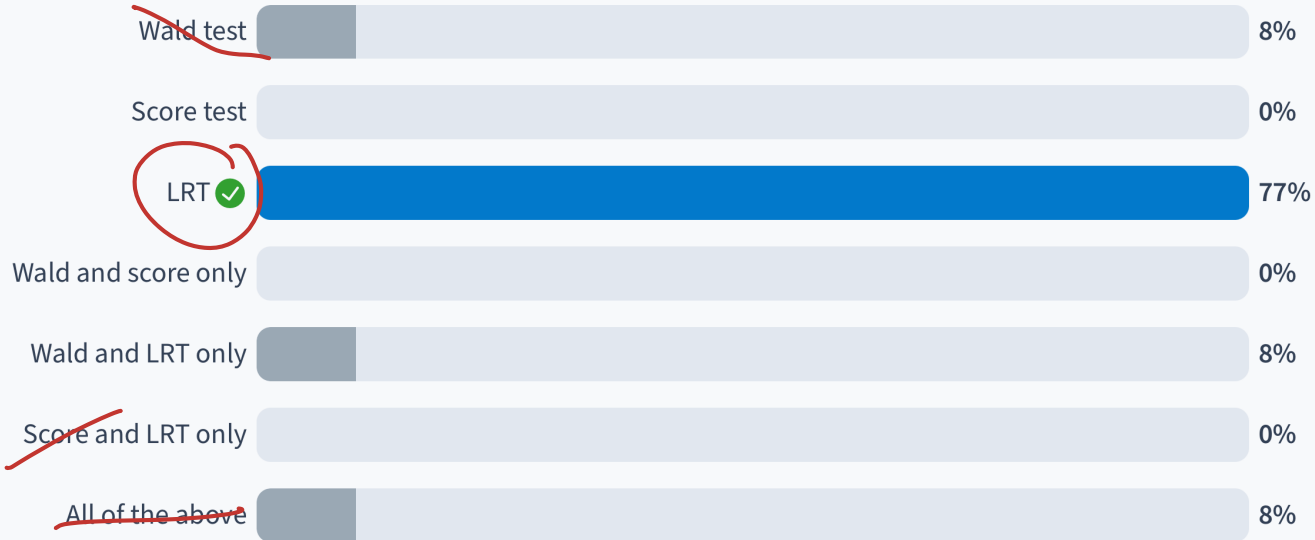
41%



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Which test(s) can be used to compare two models with different (but nested) covariates?



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# Tests and what they're used for

	<u>Wald test</u>	<u>Score test</u>	<u>LRT</u>
Used to test significance of single coefficient	✓	✓	✓
<u>Can be</u> used to report confidence interval for a single coefficient	✓	✓	✓
Confidence interval reported by R for a single coefficient (and <u>most commonly used</u> )	✓	✗	✗
Use to test significance/contribution to outcome prediction of <u>multi-level categorical covariate</u>	✗	✗	✓
Used for comparing two models with different (but nested) <u>covariates</u>	✗	✗	✓

# So how would the Wald test and LRT show up in research?

- Wald test
  - Often used when reporting estimates
  - Generally presented using a forest plot or table of ORs or RRs
    - Then we highlight the specific variable of interest in text
    - Will include the OR/RR estimate (not the coefficient like we saw today) with the 95% CI and proper interpretation of result
  
- LRT
  - Often when performing model selection and comparing two models
    - Reporting model selection Cross Validated post
  - Often does not show up explicitly in our reports, but is essential to get to our final model!!