Lesson 3: Language of Probability

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2025-10-06

Learning Objectives

1. Use set notation, Venn diagrams, and the concepts of unions, intersections, complements, and mutually exclusive events to represent and describe events.
2. Apply the axioms of probability and related properties to calculate probabilities and prove simple results.
3. Explain and use De Morgan's Laws to simplify and solve probability problems.
4. Connect partitions and all rules of probability to calculate probabilities.

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Simulations
- Probability axioms
- Probability properties
- Counting
- Rules of probability

Probability for random variables

- Probability functions: pmfs/pdfs
- Cumulative distribution functions (CDFs)
- Important distributions

Two random variables

- Transformations
- Joint distributions
- Independence and conditioning

Expected values

- Means / expected values
- Linear combinations and conditioning
- Variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

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Set Theory (1/2)

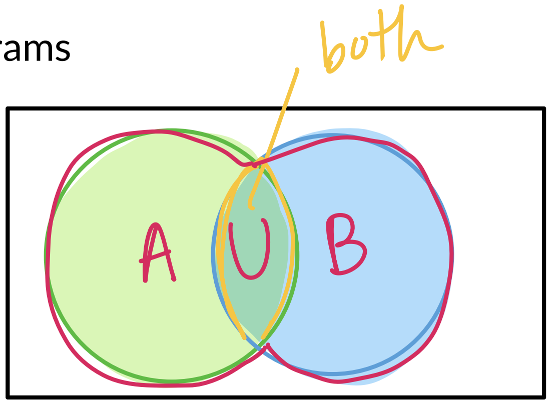
Definition: Union

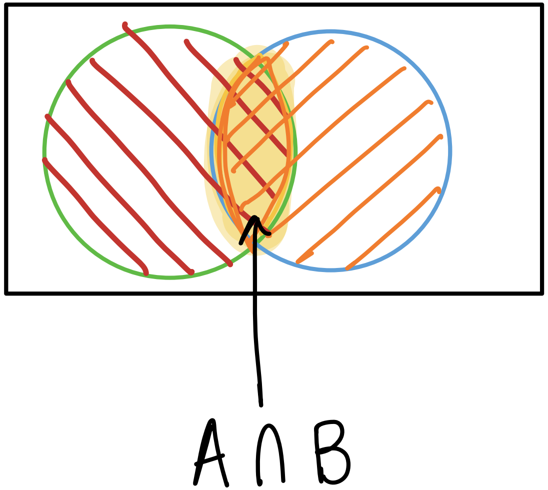
The **union** of events A and B, denoted by $A \cup B$, contains all outcomes that are in A *or* B or both

Definition: Intersection

The **intersection** of events A and B, denoted by $A \cap B$, contains all outcomes that are both in A *and* B.

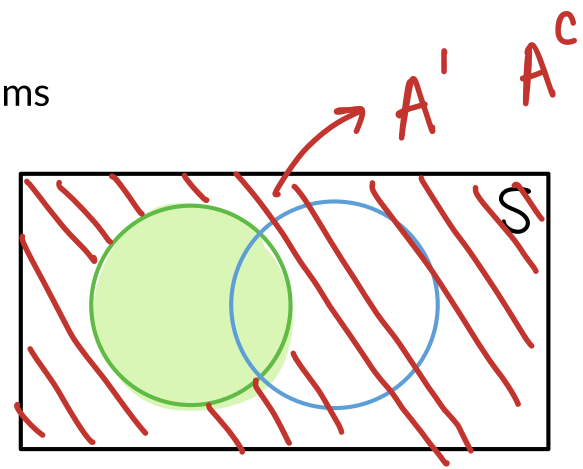
Venn diagrams

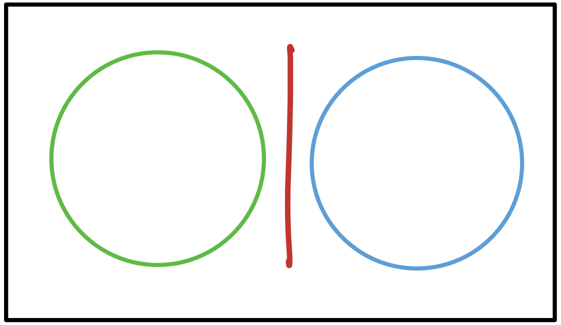




Set Theory (2/2)

Venn diagrams





$A \cap B = \phi$

Definition: Complement

The **complement** of event A, denoted by A^C or A', contains all outcomes in the sample space S that are *not* in A.

Definition: Mutually Exclusive

Events A and B are **mutually exclusive**, or disjoint, if they have no outcomes in common. In this case $A \cap B = \emptyset$, where $\emptyset$ is the empty set.

How can we code some of these? (1/2)

Example: Simulating Two Rolls of a Fair Four-Sided Die

We're going to roll two four-sided dice. **This time, let's say event A is rolling matching numbers and event B is rolling at least one 2.**

- First, we simulate rolling two four-sided dice 10,000 times

```r
set.seed(1002)
rolls = replicate(10000, sample(x = 1:4, size = 2, replace = TRUE))
```

Reps

	1	2	3 ... 10000
roll 1	1	2	4
roll 2	1	3	2
	T	F	F

- Now, we can create logical vectors for events A and B

```r
event_A = ( rolls[1, ] == rolls[2, ] )
head(event_A, 10)
```

```
[1] FALSE FALSE FALSE FALSE FALSE FALSE  TRUE FALSE FALSE FALSE
```

```r
event_B = ( rolls[1, ] == 2 | rolls[2, ] == 2 )
head(event_B, 10)
```

```
[1] FALSE FALSE FALSE  TRUE FALSE  TRUE FALSE  TRUE FALSE FALSE
```

OR

$B = C \cup D$

C first dice rolls 2

D second dice rolls a 2

How can we code some of these? (2/2)

Union

$A \cup B$

`A | B`

```
event_A_or_B = event_A | event_B
head(event_A_or_B, 10)
```

```
[1] FALSE FALSE FALSE  TRUE FALSE  TRUE  TRUE
TRUE FALSE FALSE
```

Intersection

$A \cap B$ `A & B`

```
event_A_and_B = event_A & event_B
head(event_A_and_B, 10)
```

```
[1] FALSE FALSE FALSE FALSE FALSE FALSE FALSE
FALSE FALSE FALSE
```

Complement

A^c or A' `!= A`

```
event_not_A = !event_A
event_not_B = event_B != TRUE
head(event_not_A, 10)
```

```
[1]  TRUE  TRUE  TRUE  TRUE  TRUE  TRUE FALSE
TRUE  TRUE  TRUE
```

Mutually Exclusive

$A \cap B = \emptyset$ `A & B == NA`

```
sum(event_A_and_B == TRUE)
```

```
[1] 621
```

0

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Probability Axioms

Axiom 1

For every event A, $0 \le \mathbb{P}(A) \le 1$. Probability is between 0 and 1.

Axiom 2

For the sample space S, $\mathbb{P}(S) = 1$.

Axiom 3

If $A_1, A_2, A_3, \ldots$, is a collection of **disjoint** events, then

A_∞

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

$A_1 \cap A_2 = \emptyset$

The probability of at least one A_i is the sum of the individual probabilities of each.

$$P(A) = \frac{|A|}{|S|}$$

$$P(S) = \frac{|S|}{|S|} = 1$$

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_\infty) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_\infty)$$

$$P(A \cup B) = P(A) + P(B)$$

only if disjoint/mutually exc.

Some probability properties

Using the Axioms, we can prove all other probability properties! Events A, B, and C are not necessarily disjoint!

Proposition 1

For any event A, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

Proposition 2

$\mathbb{P}(\emptyset) = 0$

Proposition 3

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

Proposition 4

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

where A and B are not necessarily disjoint

Proposition 5

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$

Proposition 1 Proof

Proposition 1

For any event A, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

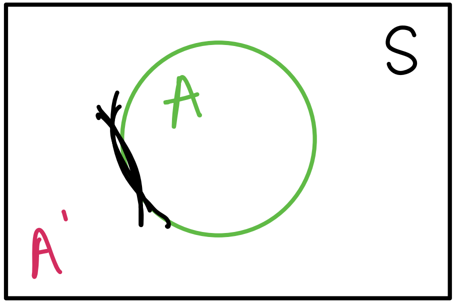
Use Axioms!

A1: $0 \leq \mathbb{P}(A) \leq 1$

A2: $\mathbb{P}(S) = 1$

A3: For disjoint A_i,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$



A & A' are disjoint

$$A \cup \underline{A'} = S$$

$$P(A \cup A') = \underbrace{P(S)}_{=1}$$

$$P(A \cup A') = 1$$

$$\downarrow$$

$$P(A) + P(A') = 1$$

$$-P(A') \qquad -P(A')$$

$$P(A) = 1 - P(A')$$

Proposition 2 Proof

Proposition 2

$\mathbb{P}(\emptyset) = 0$

Use Axioms!

A1: $0 \leq \mathbb{P}(A) \leq 1$

A2: $\mathbb{P}(S) = 1$

A3: For disjoint A_i,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

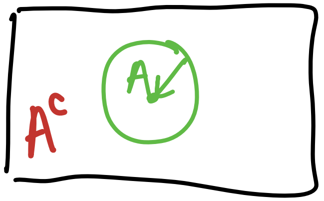
prop 1: $P(A) = 1 - P(A^c)$

let $A = \phi \qquad A^c = S$

$P(\phi) = 1 - P(S)$

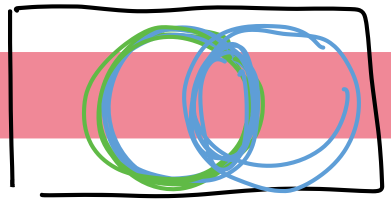
$\hookrightarrow = 1$

$P(\phi) = 1 - 1 = 0$



Proposition 3 Proof

$P(A)$ vs $P(A \cap B)$



Proposition 3

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

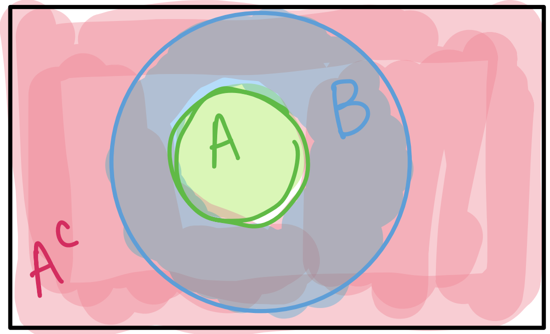
Use Axioms!

A1: $0 \leq \mathbb{P}(A) \leq 1$

A2: $\mathbb{P}(S) = 1$

A3: For disjoint A_i,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$



$$B = A \cup (B \cap A^c)$$

b/c disjoint

$$P(B) = P(A) + \underbrace{P(B \cap A^c)}_{\geq 0}$$

A & A^c are disjoint

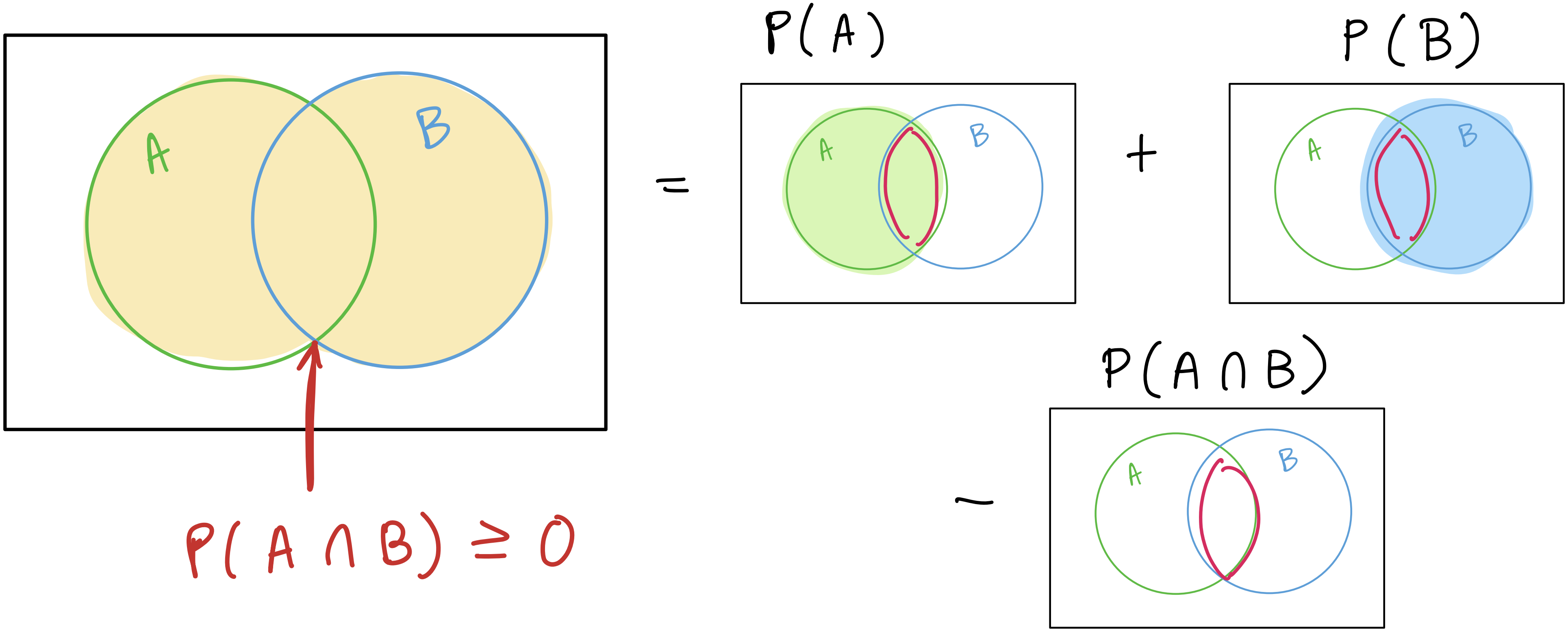
$(B \cap A^c) \subseteq A^c$

$$P(B) \geq P(A)$$

Proposition 4 Visual Proof

Proposition 4

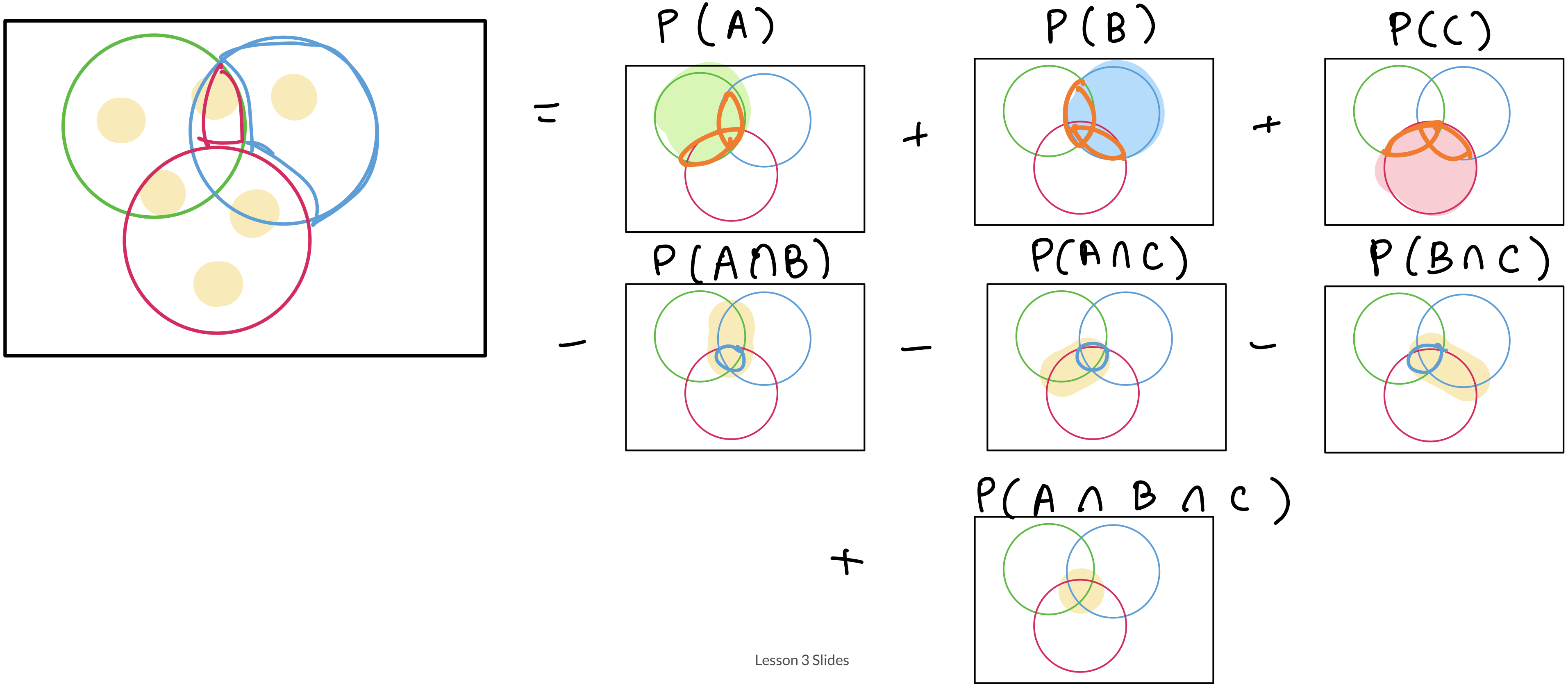
$$\underline{\mathbb{P}(A \cup B)} = \underline{\mathbb{P}(A) + \mathbb{P}(B)} - \underline{\mathbb{P}(A \cap B)}$$



Proposition 5 Visual Proof

Proposition 5

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$



Some final remarks on these proposition

- Notice how we spliced events into multiple **disjoint** events
  - It is often easier to work with disjoint events

- If we want to calculate the probability for one event, we may need to get creative with how we manipulate other events and the sample space
  - Helps us use any incomplete information we have

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De Morgan's Laws

$$\sum_{i=1}^{n} P(A_i) = P(A_1) + P(A_2) + \dots + P(A_n)$$

$$\bigcap_{i=1}^{n} B_i = B_1 \cap B_2 \cap B_3 \cap \dots \cap B_n$$

Theorem: De Morgan's 1st Law

For a collection of events (sets) $A_1, A_2, A_3, \dots$

$$\bigcap_{i=1}^{n} A_i^C = \left(\bigcup_{i=1}^{n} A_i\right)^C$$

"all not A = (at least one event A)^C" or "intersection of the complements is the complement of the union"

Theorem: De Morgan's 2nd Law

For a collection of events (sets) $A_1, A_2, A_3, \dots$

$$\bigcup_{i=1}^{n} A_i^C = \left(\bigcap_{i=1}^{n} A_i\right)^C$$

"at least one event not A = (all A)^C" or "union of complements is complement of the intersection"

BP example variation (1/3)

- Suppose you have n subjects in a study.
- Let H_i be the event that person i has high BP, for $i = 1 \ldots n$.

Use set theory notation to denote the following events:

1. Event subject i does not have high BP
2. Event all n subjects have high BP
3. Event at least one subject has high BP
4. Event all of them do not have high BP
5. Event at least one subject does not have high BP

BP example variation (2/3)

- Suppose you have n subjects in a study.
- Let H_i be the event that person i has high BP, for $i = 1 \ldots n$.

Use set theory notation to denote the following events:

1. Event subject i does not have high BP

H_i' or H_i^c

2. Event all n subjects have high BP

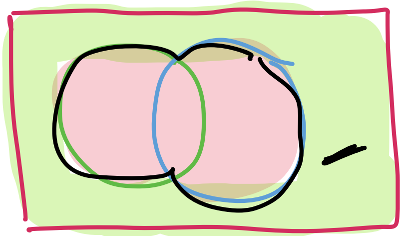
P1 & P2: $H_1 \cap H_2 \xrightarrow{\text{to } n} H_1 \cap H_2 \cap H_3 \cap \ldots \cap H_n = \bigcap_{i=1}^{n} H_i$

3. Event at least one subject has high BP

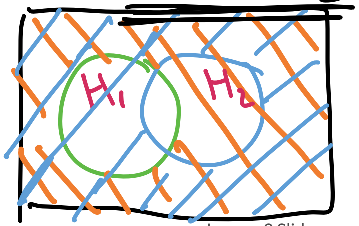
P1 & P2: $H_1 \cup H_2$

$H_1 \cup H_2 = \left(H_1^c \cap H_2^c\right)^c$

$\bigcup_{i=1}^{n} H_i = \left(\bigcap_{i=1}^{n} H_i^c\right)^c$

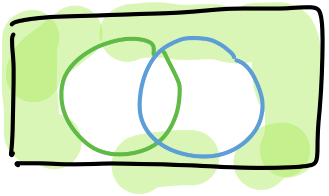


$H_1^c \cap H_2^c$



BP example variation (3/3)

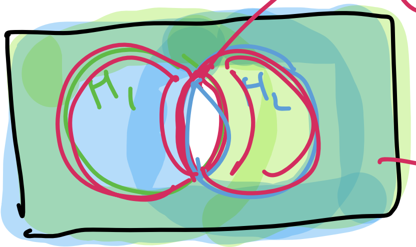
4. Event all of them do not have high BP



$$\bigcap_{i=1}^{n} H_i^c$$

5. Event at least one subject does not have high BP

P 1 P 2



$(H_1 \cap H_2)^c$

$H_1^c \cup H_2^c$

everything but
Both having high BP

$$(H_1 \cap H_2 \cap H_3 \cap \dots \cap H_n)^c = \left(\bigcap_{i=1}^{n} H_i\right)^c = \left(\bigcup_{i=1}^{n} H_i^c\right)$$

Remarks on De Morgan’s Laws

- These laws also hold for infinite collections of events.
- Draw Venn diagrams to convince yourself that these are true!
- These laws are *very* useful when calculating probabilities.
  - This is because calculating the probability of the **intersection of events is often much easier than the union of events.**
  - This is not obvious right now, but we will see in the coming chapters why.

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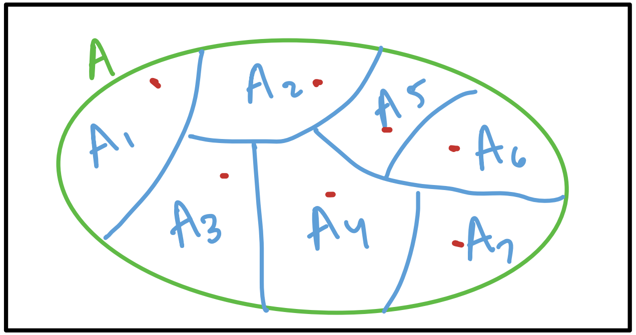
Partitions

$A_1, A_2, A_3, \dots, A_n$



Definition: Partition

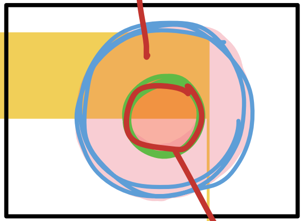
A set of events $\{A_i\}_{i=1}^{n}$ create a **partition** of A, if

- the A_i's are disjoint (mutually exclusive) and
- $\bigcup_{i=1}^{n} A_i = A$

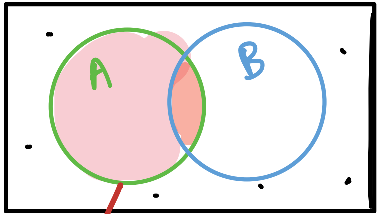
~~Example 2~~ Note

- If $A \subset B$, then $\{A, B \cap A^C\}$ is a set of partition of B.
- If $S = \bigcup_{i=1}^{n} A_i$, and the A_i's are disjoint, then the A_i's are a partition of the sample space.

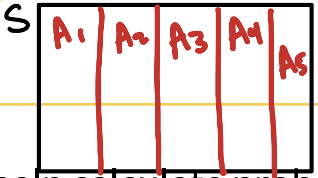
$B \cap A^c$



A



$A = (A \cap B^c) \cup (B \cap A)$



Creating partitions is sometimes used to help calculate probabilities, since by Axiom 3 we can add the probabilities of disjoint events.

Weekly medications

Example 3

If a subject has an

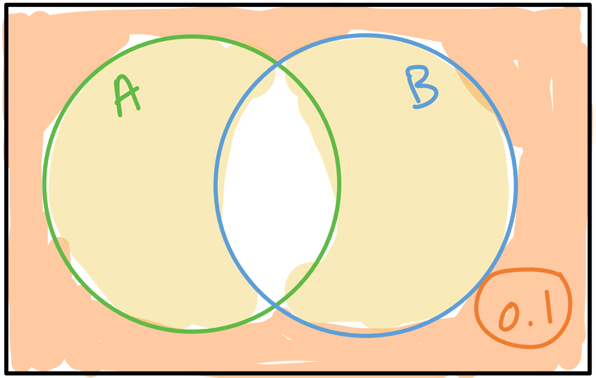
- 80% chance of taking their medication *this* week, $P(A) = 0.8$
- 70% chance of taking their medication *next* week, and $P(B) = 0.7$
- 10% chance of *not* taking their medication *either* week, $P((A \cup B)^c) = 0.1$

then find the probability of them taking their medication exactly one of the two weeks.

Hint: Draw a Venn diagram labelling each of the parts to find the probability.

Let A = take med this week
B = take med next week

	THIS Med	THIS N/M
NEXT Med		
NEXT N/Med		



De Morgan

$P(A^c \cap B^c) = 0.1$

GOAL

$P(A \cap B^c) + P(B \cap A^c)$ OR $P(A \cup B) - P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B) = 1 - P((A \cup B)^c)$
$= 1 - 0.1$

$= 0.9$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$0.9 = \underbrace{0.8 + 0.7}_{1.5} - P(A \cap B)$$

$$\begin{array}{ccc} -0.9 & & \\ +P(A \cap B) & 1.5 - 0.9 & + P(A \cap B) \end{array}$$

$$P(A \cap B) = 0.6$$

$$P(A \cup B) - P(A \cap B) = 0.9 - 0.6$$

$$= 0.3$$

The prob of take their med <u>exactly</u> one week is 0.3.