Lesson 4: Rules of probability

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Learning objectives

1. Define independence of 2-3 events and decide whether two or more events are independent
2. Define key facts for conditional probabilities and calculate conditional probabilities.
3. Calculate the probability of an event using Bayes' Theorem, Higher Order Multiplication Rule, and the Law of Total Probabilities

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Simulations
- Probability axioms
- Probability properties
- Counting ← next
- Rules of probability

Probability for random variables

- Probability functions: pmfs/pdfs
- Cumulative distribution functions (CDFs)
- Important distributions

Two random variables

- Transformations
- Joint distributions
- Independence and conditioning

Expected values

- Means / expected values
- Linear combinations and conditioning
- Variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

General Process for Probability Word Problems

1. Clearly define your events of interest & mathematically define goal prob.
2. Translate question to probability using defined events OR Venn Diagram
3. Ask yourself:
   - Are we sampling with or without replacement?
   - Does order matter? / distinguishable?
4. Use axioms, properties, partitions, facts, etc. to define the end probability calculation into smaller parts
   - If probabilities are given to you, Venn Diagrams may help you parse out the events and probability calculations
   - If you need to find probabilities with counting, pictures or diagrams might help here
5. Write out a concluding statement that gives the probability context
6. (For own check) Make sure the calculated probability follows the axioms. Is is between 0 and 1?

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Independent Events

Definition: Independence

from weekly med

Events A and B are **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

$P(A \cap B) = 0.6$
$\stackrel{?}{=} 0.8 \cdot 0.7$
$\neq 0.56$

Notation: For shorthand, we sometimes write $A \perp\!\!\!\perp B$, to denote that A and B are independent events.

$\perp$

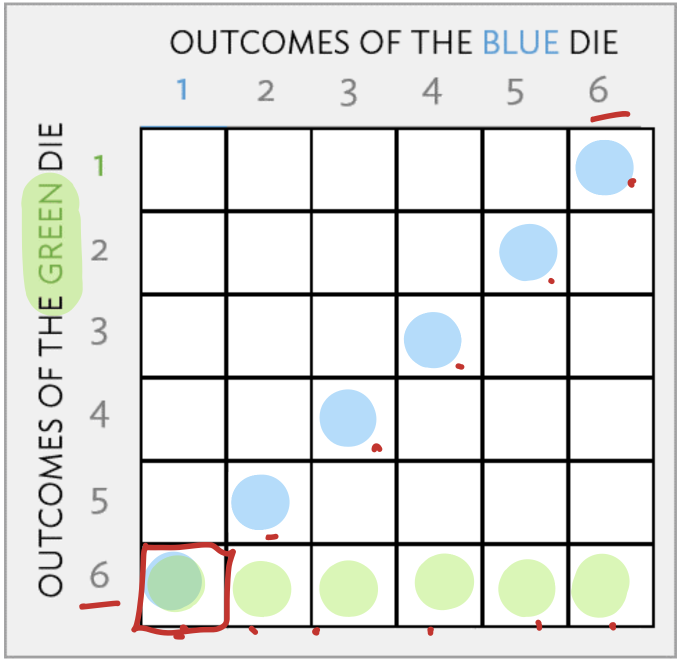
- Also note:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) \implies A \perp\!\!\!\perp B$$
$$A \perp\!\!\!\perp B \implies \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Example of two dice

Example 1

Two dice (green and blue) are rolled. Let $A =$ event a total of 7 appears, and $B =$ event green die is a six. Are events A and B independent?



$$P(A \cap B) \stackrel{?}{=} P(A)\,P(B)$$

$$\star\; P(A) = \frac{|A|}{|S|}$$

$$\frac{1}{6 \cdot 6} = \frac{1}{36} \qquad \frac{6}{36} = \frac{1}{6} \qquad \frac{1}{6}$$

$$\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

$$\frac{1}{36} = \frac{1}{36}$$

$$\Rightarrow \; A \perp B$$

"thus"

Example of two dice: simulating in R (1/2)

Example 1

Two dice (green and blue) are rolled. Let A = event a total of 7 appears, and B = event green die is a six. Are events A and B independent?

6-sided die

roll 2x

```
set.seed(1002)
reps = 10000
rolls = replicate(reps, sample(x = 1:6, size = 2, replace = TRUE))
rolls[, 1:10]
```

1st 10 rolls

```
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
[1,]    4    5    5    5    6    5    2    4    3     4
[2,]    1    4    6    3    1    1    5    5    6     3
```

```
event_A = ( rolls[1, ] + rolls[2, ] == 7 )
head(event_A, 10)
```

```
[1] FALSE FALSE FALSE FALSE  TRUE FALSE  TRUE FALSE FALSE  TRUE
```

```
event_B = ( rolls[1, ] == 6 )
head(event_B, 10)
```

```
[1] FALSE FALSE FALSE FALSE  TRUE FALSE FALSE FALSE FALSE FALSE
```

Example of two dice: simulating in R (1/2)

Example 1

Two dice (green and blue) are rolled. Let A = event a total of 7 appears, and B = event green die is a six. Are events A and B independent?

$P(A)$ $P(B)$

```
( sum(event_A) / reps ) * ( sum(event_B) / reps )
```

```
[1] 0.0286608
```

```
event_A_and_B = ( rolls[1, ] + rolls[2, ] == 7 ) & ( rolls[1, ] == 6 )
head(event_A_and_B, 10)
```

```
 [1] FALSE FALSE FALSE FALSE  TRUE FALSE FALSE FALSE FALSE FALSE
```

```
sum(event_A_and_B) / reps
```

```
[1] 0.0284
```

$\rightarrow P(A \cap B)$

$P(A \cap B) = P(A)P(B)$

Independence of 3 Events

Definition: Independence of 3 Events

Events A, B, and C are *mutually* **independent** if

1. • $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$
   • $\mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C)$
   • $\mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C)$
2. $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$

$A \perp B \perp C$

$$\left.\begin{matrix} A \perp B \\ A \perp C \\ B \perp C \end{matrix}\right\} \text{mutually}$$

Remark:

On your homework you will show that $(1) \not\Rightarrow (2)$ and $(2) \not\Rightarrow (1)$.

Probability at least one smoker

Example 2

Suppose you take a random sample of n people, of which people are smokers and non-smokers independently of each other. Let

- $A_i =$ event person i is a smoker, for $i = 1, \ldots, n$, and
- $p_i =$ probability person i is a smoker, for $i = 1, \ldots, n$.

Find the probability that at least one person in the random sample is a smoker.

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Conditional Probability facts (1/2)

Fact 1: General Multiplication Rule

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B|A)$$

$$P(A \cap B) = P(B)\, P(A|B)$$

Fact 2: Conditional Probability Definition

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

A given B

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

if $A \perp B$

$$P(A|B) = P(A)$$

$$= \frac{P(A \cap B)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}}$$

Conditional Probability facts (2/2)

$$P(B|A) + P(B^c|A) = 1$$

Fact 3

If A and B are independent events ($A \perp\!\!\!\perp B$), then

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$

$$P(B|A) = P(B)$$

Fact 4

$\mathbb{P}(A|B)$ is a probability, meaning that it satisfies the probability axioms. In particular,

$$\mathbb{P}(A|B) + \mathbb{P}(A^C|B) = 1$$

if $A \perp B$

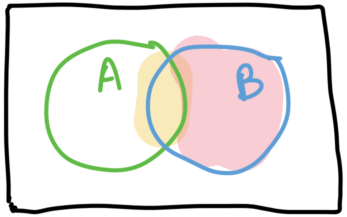
$$P(A|B) = P(A)$$

$$\hookrightarrow = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}}$$

$$P(A|B) + P(A^c|B) = 1$$

$$\frac{P(A \cap B)}{P(B)} + \frac{P(A^c \cap B)}{P(B)} = 1 \quad \times P(B)$$

$$P(A \cap B) + P(A^c \cap B) = P(B)$$

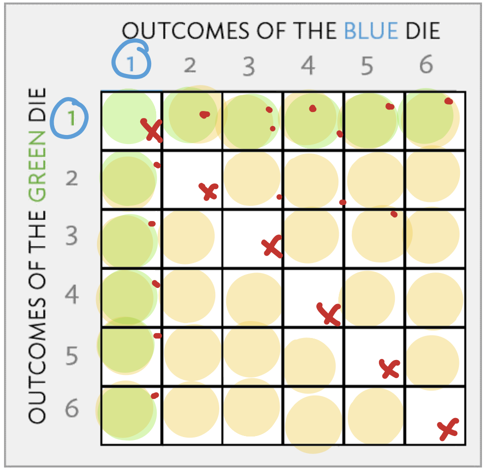


Conditional probability with two dice

① Let A = one die is a 1
B = dice do not show same face

Example 3

Two dice (green and blue) are rolled. If the dice do not show the same face, what is the probability that one of the dice is a 1?



$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{10/\cancel{36}}{30/\cancel{36}} = \frac{1}{3}$$

The prob of 1 die being a 1 given 2 dice do not match is 1/3.

Conditional probability with two dice: simulations

Example 3

Two dice (green and blue) are rolled. If the dice do not show the same face, what is the probability that one of the dice is a 1?

```r
set.seed(1002)
rolls = replicate(reps, sample(x = 1:6, size = 2, replace = TRUE))
rolls[, 1:10]
```

```
     [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
[1,]    4    5    5    5    6    5    2    4    3     4
[2,]    1    4    6    3    1    1    5    5    6     3
```

```r
event_A = ( rolls[1, ] != rolls[2, ] )
head(event_A, 10)
```

```
 [1] TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE TRUE
```

```r
event_B = ( rolls[1, ] == 1 | rolls[2, ] == 1 )
head(event_B, 10)
```

```
 [1]  TRUE FALSE FALSE FALSE  TRUE  TRUE FALSE FALSE FALSE FALSE
```

$$P(A \cap B) = \frac{sum(A \& B)}{\cancel{reps}}$$

$$P(A) = \frac{sum(A)}{\cancel{reps}}$$

```r
sum(event_B & event_A) / sum(event_A)
```

```
[1] 0.3315328
```

$\underbrace{sum(event\_B \,\&\, event\_A)}_{P(A \cap B)} \; / \; \underbrace{sum(event\_A)}_{P(A)}$

Monty Hall Problem

Survivor Season 42

With the Wiki page on it!

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Bayes' Rule for two events

- We can use the conditional probability ($\mathbb{P}(A|B)$) to get information on the flipped conditional probability ($\mathbb{P}(B|A)$)

Theorem: Bayes' Rule (for two events)

For any two events A and B with nonzero probabilties,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A) \cdot \mathbb{P}(B|A)}{\mathbb{P}(B)}$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A)P(B|A)$$

Calculating probability with Higher Order Multiplication Rule

Example 4 each suit 13 cards

Suppose we draw 5 cards from a standard shuffled deck of 52 cards. What is the probability of a flush, that is all the cards are of the same suit (including straight flushes)?

Higher Order Multiplication Rule

$$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2|A_1) \cdot \mathbb{P}(A_3|A_1A_2) \ldots \cdot \mathbb{P}(A_n|A_1A_2 \ldots A_{n-1})$$

$P(A \cap B) = P(A)P(B|A)$

① A_1 = card of any suit

A_i = card w/ same suit as A_1

$i = 2,3,4,5$

② $P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)$

③ order matters, no replacement

④ $P(A_1) = \frac{52}{52}$ $P(A_2|A_1) = \frac{12}{51}$

$P(A_3|A_1, A_2) = \frac{11}{50}$ $P(A_4|A_1, A_2, A_3) = \frac{10}{49}$

$P(A_5|A_1, A_2, A_3, A_4) = 9/48$

$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = \underbrace{P(A_1)P(A_2|A_1)}_{P(A_1 \cap A_2)} \cdot P(A_3|A_1, A_2) \cdot P(A_4|A_1, A_2, A_3) \cdot P(A_5|A_1, A_2, A_3, A_4)$

$= \frac{52}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48}$

$= 0.00198$

Calculating probability with Law of Total Probability

Example 5

Suppose 1% of people assigned female at birth (AFAB) and 5% of people assigned male at birth (AMAB) are color-blind. Assume person born is equally likely AFAB or AMAB (not including intersex). What is the probability that a person chosen at random is color-blind?

$\rightarrow$ GOAL: $P(B)$

Law of Total Probability for 2 Events

For events A and B,

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^C)\\ &= \mathbb{P}(B|A)\cdot\mathbb{P}(A) + \mathbb{P}(B|A^C)\cdot\mathbb{P}(A^C)\end{aligned}$$

$P(CB \mid AFAB) = 0.01 = P(B|A)$

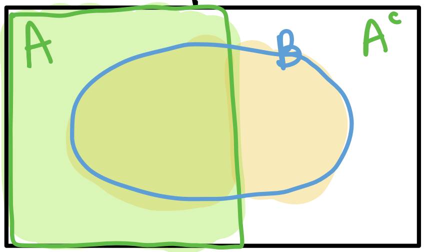
$P(CB \mid AMAB) = 0.05 = P(B|A^c)$

$P(AFAB) = 0.5 = P(A)$

$P(AMAB) = 0.5 = P(A^c)$

① Let $A = AFAB$ $\quad A^c = AMAB$

$B =$ color blind



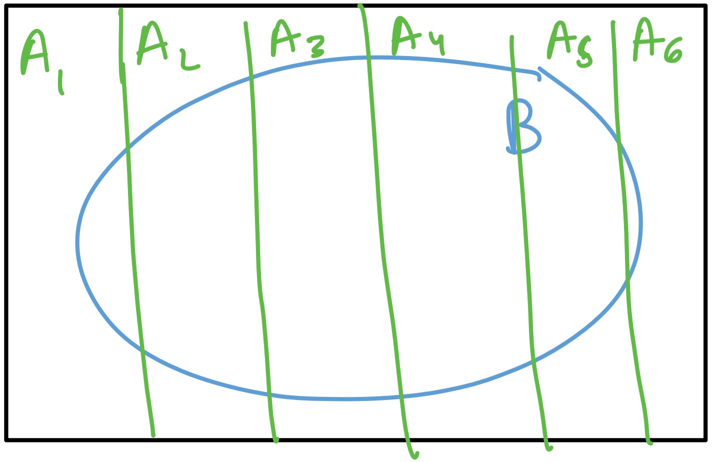
$$\begin{aligned}P(B) &= P(B|A)\,P(A) + P(B|A^c)\,P(A^c)\\ &= 0.01 \cdot 0.5 + 0.05\,(0.5)\\ &= 0.03\end{aligned}$$

General Law of Total Proability

Law of Total Probability (general)

If $\{A_i\}_{i=1}^{n} = \{A_1, A_2, \ldots, A_n\}$ form a partition of the sample space, then for event B,

$$\begin{aligned}\mathbb{P}(B) &= \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) \\ &= \sum_{i=1}^{n} \mathbb{P}(B|A_i) \cdot \mathbb{P}(A_i)\end{aligned}$$



Calculating probability with generalized Law of Total Probability

Example 3

Individuals are diagnosed with a particular type of cancer that can take on three different disease forms,* D_1, D_2, and D_3. It is known that amongst people diagnosed with this particular type of cancer,

- 20% of people will eventually be diagnosed with form D_1, → $P(D_1)$
- 30% with form D_2, and → $P(D_2)$
- 50% with form D_3. → $P(D_3)$

The probability of requiring chemotherapy (C) differs among the three forms of disease:

- 80% with D_1, → $P(C|D_1)$
- 30% with D_2, and → $P(C|D_2)$
- 10% with D_3. → $P(C|D_3)$

Based solely on the preliminary test of being diagnosed with the cancer, what is the probability of requiring chemotherapy (the event C)?

① Event notation in problem

② $P(C)$?

③ replication and/or order matter? only considering 1 case so N/A

④ labelled parts in problem

Total prob law:

$$P(C) = P(D_1 \cap C) + P(C \cap D_2) + P(C \cap D_3)$$

$$P(C) = P(D_1)P(C|D_1) + P(D_2)P(C|D_2) + P(D_3)P(C|D_3)$$

$$P(C) = 0.2 \cdot 0.8 + 0.3 \cdot 0.3 + 0.5 \cdot 0.1$$

$$= 0.16 + 0.09 + 0.05$$

$$= 0.3$$

⑤ The probability of requiring chemotherapy if you are diagnosed with cancer is 0.3

Calculate probability with both rules

$P(B \mid A) + P(B^c \mid A) = 1$
$P(A \mid B) + P(A^c \mid B) = 1$

Example 6

Suppose

- 1% of people who are AFAB aged 40-50 years have breast cancer, $P(B) = 0.01$
- an AFAB person with breast cancer has a 90% chance of a positive test from a mammogram, and $P(A \mid B) = 0.9$
- an AFAB person has a 10% chance of a false-positive result from a mammogram. $P(A \mid B^c) = 0.1$

$P(B^c) = 1 - P(B) = 0.99$

What is the probability that an AFAB person has breast cancer given that they just had a positive test?

$P(B \mid A)$

B = breast cancer

B^c = no breast cancer

A = positive result mammo

A^c = negative result mammo

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(B)\,P(A \mid B)}{P(A)}$$

Bayes' Rule

$$= \frac{P(B)\,P(A \mid B)}{P(A \cap B) + P(A \cap B^c)}$$

Law of total prob

$$= \frac{P(B)\,P(A \mid B)}{P(B)\,P(A \mid B) + P(B^c)\,P(A \mid B^c)}$$

mult rule

$$= \frac{(0.01)(0.9)}{0.01 \cdot 0.9 + 0.99(0.1)} = \boxed{0.0833}$$

The probability that an AFAB person has BC given a pos. mammo is 0.0833

Bayes' Rule

Theorem: Bayes' Rule

If $\{A_i\}_{i=1}^n$ form a partition of the sample space S, with $\mathbb{P}(A_i) > 0$ for $i = 1 \ldots n$ and $\mathbb{P}(B) > 0$, then

$$\mathbb{P}(A_j|B) = \frac{\mathbb{P}(B|A_j) \cdot \mathbb{P}(A_j)}{\sum_{i=1}^n \mathbb{P}(B|A_i) \cdot \mathbb{P}(A_i)}$$