Lesson 10: Transformations

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Learning Objectives

1. Find the pdf of a linear rescaling of a random variable
2. Find the pdf of a nonlinear transformation of a random variable using the CDF method

Distributions of transformations of random variables

- Often make transformations of RVs
- A function of a random variable is a random variable
  - If X is a random variable and g is a function then $Y = g(X)$ is a random variable
  - Since $g(X)$ is a random variable it has a distribution
- Distribution of $g(X)$ will have a different shape than the distribution of X
- Two types:
  - Linear rescalings: $g(u) = a + bu$
  - Nonlinear transformations: e.g. $g(u) = u^2, g(u) = \log(u)$

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Linear rescaling

Definition: Linear Rescaling

A **linear rescaling** is a transformation of the form $g(u) = a + bu$, where a and b are constants

- Thus, if we have a random variable, X, then a linear rescaling of X could be $M = g(X) = a + bX$
- For example, converting temperature from Celsius to Fahrenheit using $g(u) = 32 + 1.8u$ is a linear rescaling.

Example of linear rescaling (1/4)

Example 1: Linear rescaling of U

Let U be a random variable with $f_U(u) = \dfrac{4}{15}u^3$ for $1 \leq u \leq 2$. Define $V = 1 - U$

1. What are the possible values of V?
2. Is V the same random variable as U?
3. Find $P(V \leq -0.5)$.
4. Find the pdf of V.
5. Does V have the same distribution as U?

Example of linear rescaling (2/4)

Example 1: Linear rescaling of U

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Example of linear rescaling (3/4)

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Example of linear rescaling (4/4)

Example 1: Linear rescaling of U

Let U be a random variable with $f_U(u) = \dfrac{4}{15}u^3$ for $1 \le u \le 2$. Define $V = 1 - U$

4. Find the pdf of V.
5. Does V have the same distribution as U?

Summary of linear rescaling

- A linear rescaling of a random variable does not change the basic shape of its distribution, just the range of possible values.
  - It can flip it, widen it, condense it, and/or shift it
- Remember, do NOT confuse a random variable with its distribution
  - The random variable is the numerical quantity being measured
  - The distribution is the long run pattern of variation of many observed values of the random variable

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Nonlinear transformations

- What happens when we make a **nonlinear transformation**, like a logarithmic or square root transformation?
- Nonlinear transformations do *not* necessarily preserve the distribution shape
- Examples of nonlinear transformations:
  - $g(u) = u^2$
  - $g(u) = \sqrt{u}$
  - $g(u) = \log(u)$
  - $g(u) = e^u$
  - $g(u) = \frac{1}{u}$

Finding the pdf of a transformation

- Let M be a transformation of X: $M = g(X)$
- When we have a transformation of X, M, we need to follow the **CDF method** to find the pdf of M

We follow **CDF method**:

1. Start with the pdf for X
    - aka $f_X(x)$
2. Translate the domain of X to M: find the possible values of M
3. Find the CDF of M
    - aka $F_M(m) = P(M \leq m) = P(g(X) \leq m)$
    - Will require manipulating $g(X) \leq m$ in terms of X (aka X alone on the left side)
4. Take the derivative of the CDF of M with respect to m to find the pdf of M
    - aka $f_M(m) = \dfrac{d}{dm} F_M(m)$

Example of nonlinear transformation (1/4)

Example 2: Nonlinear transformation of U

Let U be a random variable with $f_U(u) = \frac{4}{15}u^3$ for $1 \leq u \leq 2$. Define $V = \log(U)$

1. What are the possible values of V?
2. Find the CDF of V
3. Find the pdf of V

Example of nonlinear transformation (2/4)

Example 2: Nonlinear transformation of U

Let U be a random variable with $f_U(u) = \dfrac{4}{15}u^3$ for $1 \le u \le 2$. Define $V = \log(U)$

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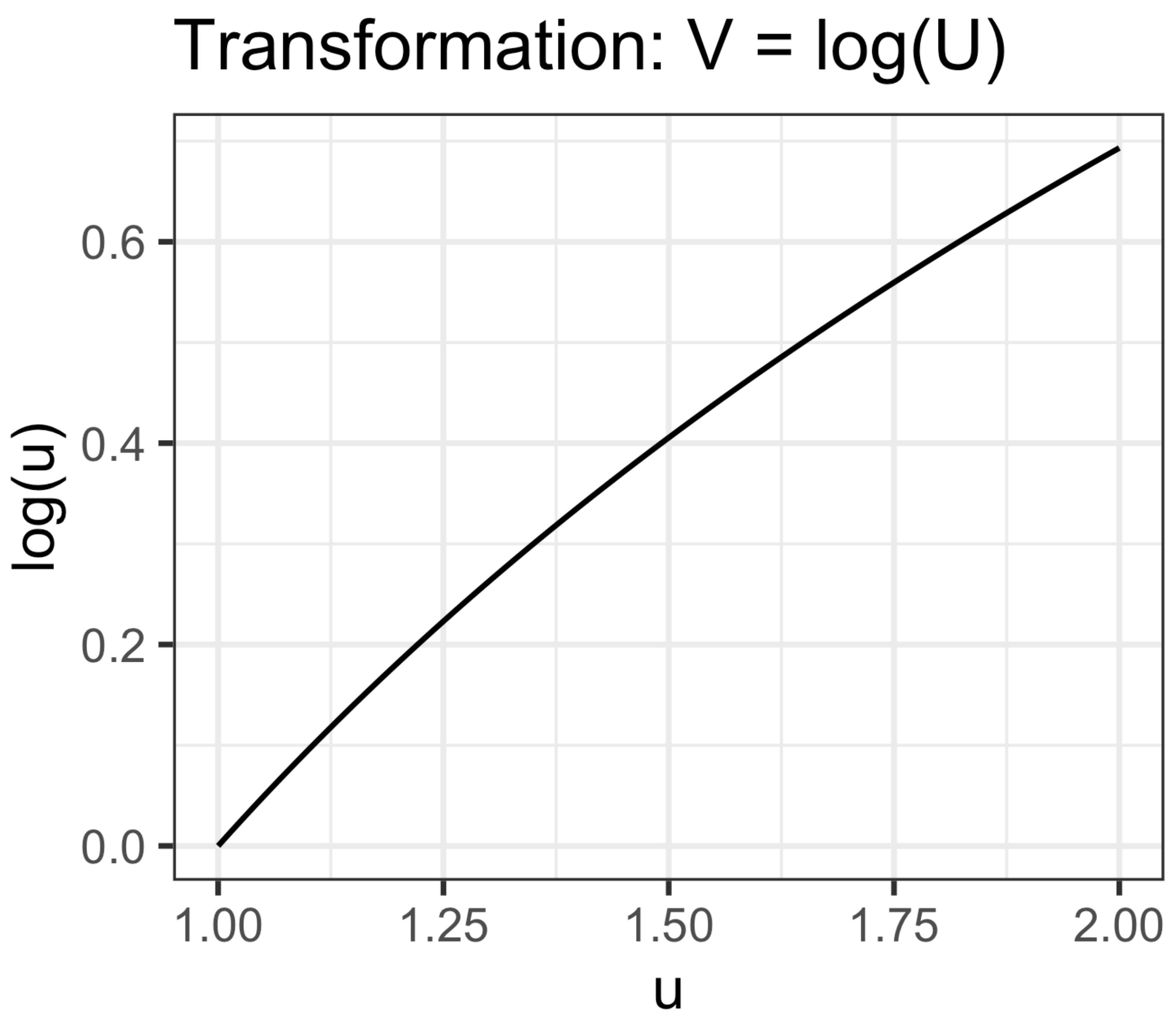

Example of nonlinear transformation (3/4)

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Example of nonlinear transformation (4/4)

Example 2: Nonlinear transformation of U

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3. Find the pdf of V

Example of nonlinear transformation: domain (1/4)

Example 3: Nonlinear transformation of X

Let X be a random variable with $f_X(x) = \dfrac{1}{2}$ for $-1 \leq x \leq 1$. Define $Y = X^2$

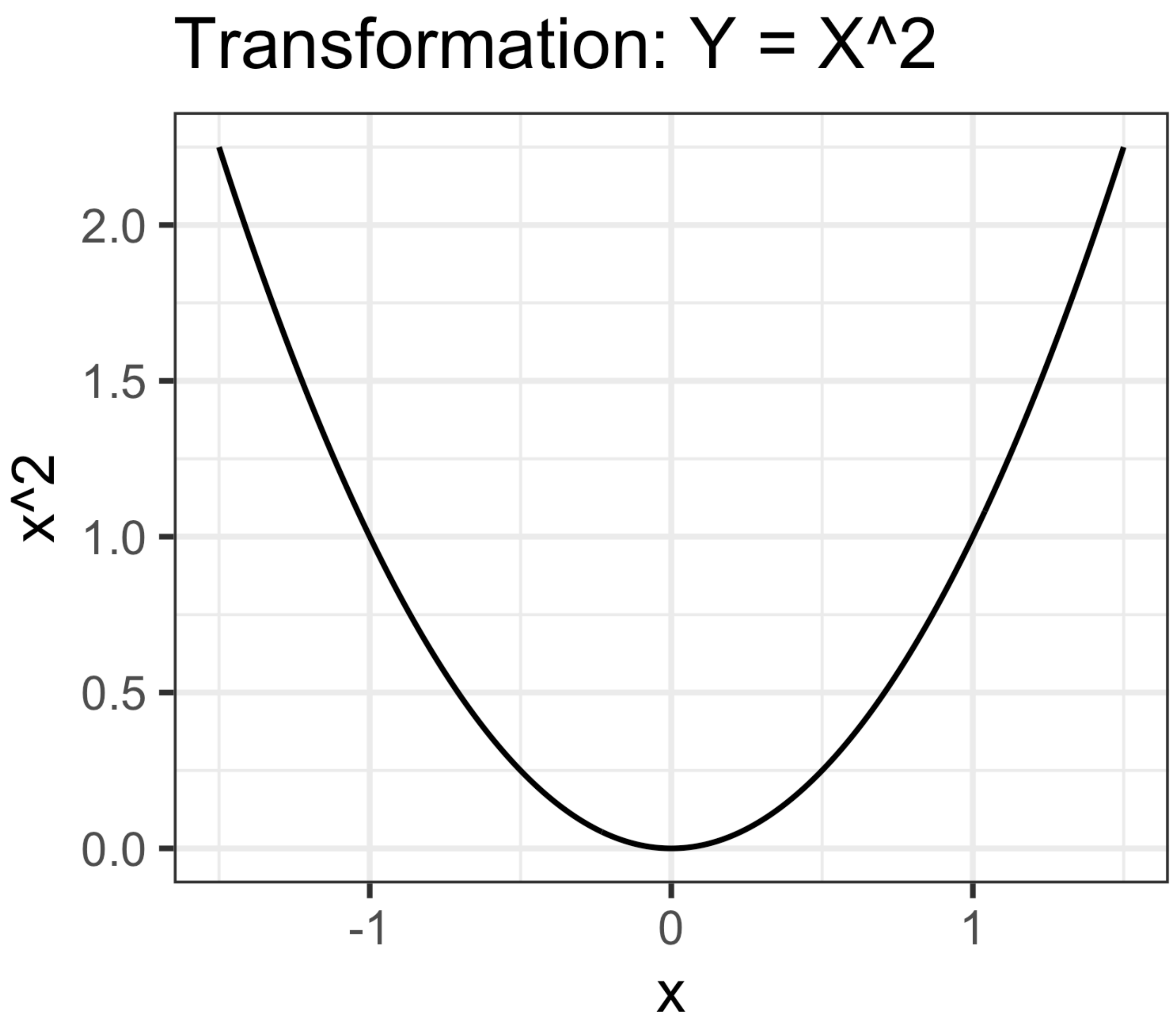
1. What are the possible values of Y?
2. Find the CDF of Y
3. Find the pdf of Y

Example of nonlinear transformation: domain (2/4)

Example 3: Nonlinear transformation of X

Let X be a random variable with $f_X(x) = \dfrac{1}{2}$ for $-1 \leq x \leq 1$. Define $Y = X^2$

1. What are the possible values of Y?



Example of nonlinear transformation: domain (3/4)

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Example of nonlinear transformation: domain (4/4)

Example 3: Nonlinear transformation of X

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3. Find the pdf of Y

Summary of nonlinear transformations

- Nonlinear transformations can change the shape of a distribution
- Always use the CDF method to find the pdf of a nonlinear transformation of a random variable
- Remember to carefully determine the possible values of the transformed random variable