Lesson 11: Joint distributions

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2025-10-29

Learning Objectives

1. Define **joint and marginal** distributions for discrete and continuous random variables
2. Calculate or find **joint and marginal** probabilities, pmf's, and CDF's for discrete random variables
3. Calculate or find **joint and marginal** probabilities, pdf's, and CDF's for continuous random variables
4. Extra practice on your own: solve double integrals in a mini lesson

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Simulations
- Probability axioms
- Probability properties
- Counting
- Rules of probability

Probability for random variables

- Probability functions: pmfs/pdfs
- Cumulative distribution functions (CDFs)
- Important distributions

Two random variables

- Transformations
- Joint distributions
- Independence and conditioning

Expected values

- Means / expected values
- Linear combinations and conditioning
- Variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

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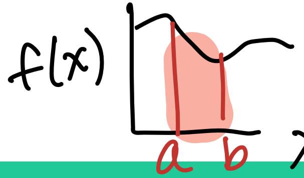
What is a joint distribution?



area under curve

Definition: joint pmf

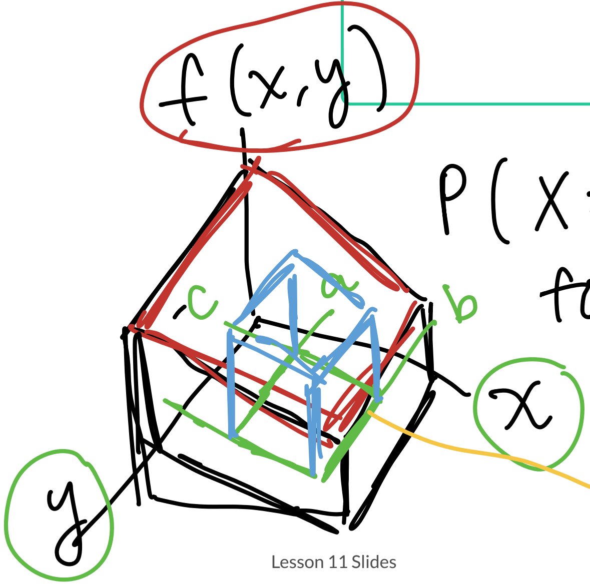
The **joint pmf** of a pair of discrete RV's X and Y is

$$\begin{aligned} p_{X,Y}(x,y) &= \mathbb{P}(X = x \cap Y = y) \\ &= \mathbb{P}(X = x, Y = y) \end{aligned}$$

Definition: joint pdf

The **joint pdf** for two continuous RVs (X and Y) is $f_{X,Y}(x,y)$, such that we have the following joint probability:

$$\mathbb{P}(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y)\,dy\,dx$$



$P(X=x, Y=y) = 0$ for cont RVs

volume w/in bounds under curve

Important properties of joint distributions

Properties of joint pmf's

- A joint pmf $p_{X,Y}(x,y)$ must satisfy the following properties:
  - $0 \geq p_{X,Y}(x,y) \leq 1$ for all x, y
  - $\sum_{\{all\ x\}} \sum_{\{all\ y\}} p_{X,Y}(x,y) = 1$

Properties of joint pdf's

- A joint pdf $f_{X,Y}(x,y)$ must satisfy the following properties:
  - $f_{X,Y}(x,y) \geq 0$ for all x, y
  - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$
- Remember that $f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$!!!

Marginal distributions

Marginal pmf's

Suppose X and Y are discrete RV's, with joint pmf $p_{X,Y}(x,y)$. Then the **marginal probability mass functions** are

marginal

$$p_X(x) = \sum_{\{all\ y\}} p_{X,Y}(x,y)$$

$$p_Y(y) = \sum_{\{all\ x\}} p_{X,Y}(x,y)$$

marg of x, sum across y

Marginal pdf's

Suppose X and Y are continuous RV's, with joint pdf $f_{X,Y}(x,y)$. Then the **marginal probability density functions** are

marginal

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

marg of x, integrate across y

Joint cumulative distribution functions (CDFs)

Joint CDF for discrete RVs

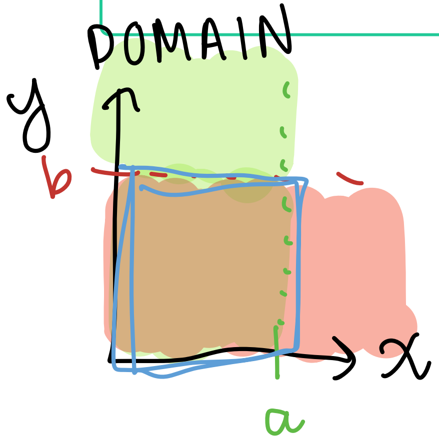
The **joint CDF** of a pair of discrete RV's X and Y is

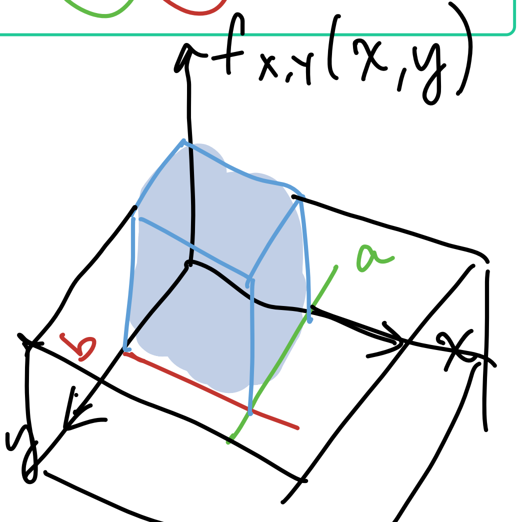
$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x \textit{ and } Y \leq y) = \mathbb{P}(X \leq x, Y \leq y)$$

Joint CDF for continuous RVs

The **joint CDF** of continuous random variables X and Y, is the function $F_{X,Y}(x,y)$, such that for all real values of x and y,

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)\,dt\,ds$$





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Joint distribution for two discrete random variables (1/5)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

1. Find $p_{X,Y}(x, y)$
2. Find $\mathbb{P}(X + Y = 3)$
3. Find $\mathbb{P}(Y = 1)$
4. Find $\mathbb{P}(Y \leq 2)$
5. Find the joint CDF $F_{X,Y}(x, y)$ for the joint pmf $p_{X,Y}(x, y)$
6. Find the marginal CDFs $F_X(x)$ and $F_Y(y)$

X = first draw

Y = second draw

Joint distribution for two discrete random variables (2/5)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

1. Find $p_{X,Y}(x, y)$
2. Find $\mathbb{P}(X + Y = 3)$

②
$$P(X+Y=3) = P(X=1, Y=2) + P(X=2, Y=1)$$
$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

①

X \ Y	1	2	3	
1	× 0	✓ 1/6	✓ 1/6	
2	✓ 1/6	× 0	✓ 1/6	
3	✓ 1/6	✓ 1/6	× 0	

$$P(X=x, Y=y) = \begin{cases} 1/6 & x \neq y \\ 0 & x = y \end{cases}$$
for $x = 1,2,3$ & $y = 1,2,3$

Joint distribution for two discrete random variables (3/5)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

3. Find $\mathbb{P}(Y = 1)$

4. Find $\mathbb{P}(Y \leq 2)$

		Y = 1	Y = 2	Y = 3
X	1	0	1/6	1/6
	2	1/6	0	1/6
	3	1/6	1/6	0
$P_Y(y)$		1/3	1/3	1/3

③ $P(Y=y) = \sum_{\text{all } x} P(X=x, Y=y)$

$P(Y=1) = \sum_{\text{all } x} P(X=x, Y=1)$

$= P(X=1, Y=1) + P(X=2, Y=1) + P(X=3, Y=1)$

$= 0 + \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

④ $P(Y \leq 2) = P(Y=1) + P(Y=2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

sum of marginal

sum of joint:

$P(Y \leq 2) = \sum_{y=1}^{2} \sum_{x=1}^{3} P(X=x, Y=y)$

Joint distribution for two discrete random variables (4/5)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

5. Find the joint CDF $F_{X,Y}(x, y)$ for the joint pmf $p_{X,Y}(x, y)$

$$P(X \le 1, Y \le 1) = P(X=1, Y=1)$$

$$P(X \le 1, Y \le 2) = P(X=1, Y=1) + P(X=1, Y=2)$$

$$P(X \le 2, Y \le 2) = \sum_{y=1}^{2} \sum_{x=1}^{2} P(X=x, Y=y)$$

jCDF

X \ Y	1	2	3
1	0	1/6	1/3
2	1/6	1/3	2/3
3	1/3	2/3	1

~~j~~PMF

X \ Y	1	2	3
1	0	1/6	1/6
2	1/6	0	1/6
3	1/6	1/6	0

Joint distribution for two discrete random variables (5/5)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

6. Find the marginal CDFs $F_X(x)$ and $F_Y(y)$

jCDF

X \ Y	1	2	3	$F_X(x)$
1	0	1/6	1/3	1/3
2	1/6	1/3	2/3	2/3
3	1/3	2/3	1	1
$F_Y(y)$	1/3	2/3	1	

$$F_X(x) = P(X \le x)$$

$$F_X(1) = P(X \le 1)$$

$$= P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$$

OR

$$= P(X=1)$$

Quick remarks on the joint and marginal CDF

- $F_X(x)$: right most columns of the CDF table (where the Y values are largest)
- $F_Y(y)$: bottom row of the table (where X values are largest)
- $F_X(x) = \lim_{y \to \infty} F_{X,Y}(x, y)$ → marginal CDF of X "sums" takes Y to its max value
- $F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x, y)$

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Common steps for joint pdfs and CDFs

1. Set up the domain of the pdf with a picture

2. Translate to needed integrands
   - For probability: shade in the area of interest, then translate
   - For expected value: translate domain

☆ transformation

3. Set up integral: $dxdy$ or $dydx$?

4. Solve integral!

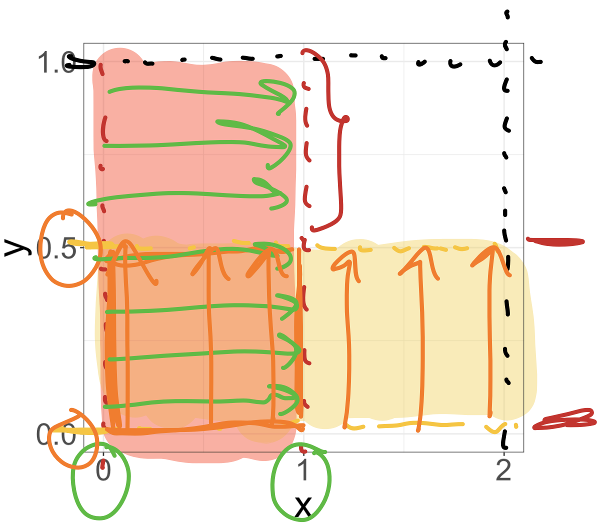
Example 2: Joint pdf (1/2)

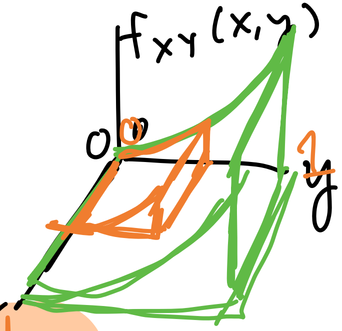
Example 2.1

Let $f_{X,Y}(x,y) = \frac{3}{2}y^2$, for $0 \le x \le 2$, $0 \le y \le 1$.

1. Find $\mathbb{P}(0 \le X \le 1, 0 \le Y \le \frac{1}{2})$

①





② $P(0 \le X \le 1, 0 \le Y \le 1/2)$

$$= \int_0^{0.5}\int_0^1 f_{X,Y}(x,y)\,dx\,dy$$

constant

$$= \int_0^{0.5}\int_0^1 \frac{3}{2}y^2\,dx\,dy$$

$$= \int_0^{0.5}\left[\frac{3}{2}y^2 \cdot x\right]_{x=0}^{x=1} dy = \int_0^{0.5} \frac{3}{2}y^2(1) - 0\,dy$$

$$= \int_0^{0.5} \frac{3}{2}y^2\,dy = \frac{1}{2}y^3\Big|_{y=0}^{y=0.5}$$

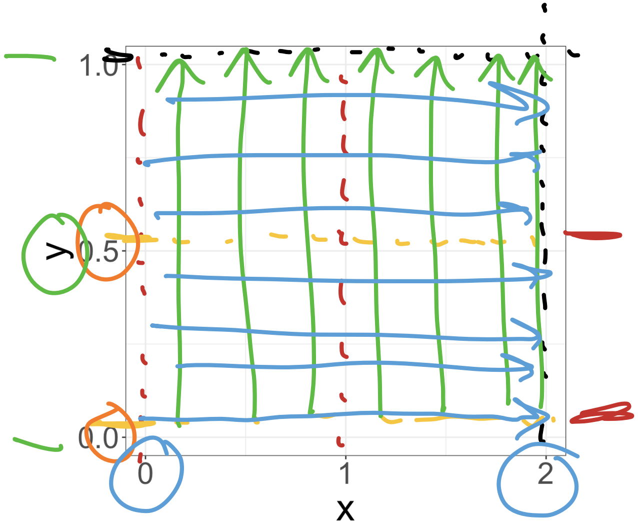
$$= \frac{1}{2}\left(\frac{1}{2}\right)^3 - \frac{1}{2}0 = \frac{1}{16}$$

Example 2: Joint pdf (2/2)

Example 2.2

Let $f_{X,Y}(x,y) = \frac{3}{2}y^2$, for $0 \le x \le 2$, $0 \le y \le 1$.

2. Find $f_X(x)$ and $f_Y(y)$.



$f(x)$: integrate out y

$$f_X(x) = \int_0^1 f_{X,Y}(x,y)\,dy = \int_0^1 \frac{3}{2}y^2\,dy$$

$$= \frac{1}{2}y^3 \Big|_{y=0}^{y=1} = \frac{1}{2} - 0 = \frac{1}{2}$$

$$f_X(x) = \frac{1}{2} \quad \text{for } 0 \le x \le 2$$

$f_Y(y)$: integrate out x

$$f_Y(y) = \int_0^2 f_{X,Y}(x,y)\,dx = \int_0^2 \frac{3}{2}y^2\,dx$$

$$= \frac{3}{2}y^2 x \Big|_{x=0}^{x=2} = \frac{3}{2}y^2(2) - 0 = 3y^2$$

$$f_Y(y) = 3y^2 \quad 0 \le y \le 1$$

Example with more complicated pdf (1/2)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Example 3.1

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

1. Find $f_X(x)$ and $f_Y(y)$.

Example with more complicated pdf (2/2)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Example 3.2

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

2. Find $\mathbb{P}(Y < 3)$.

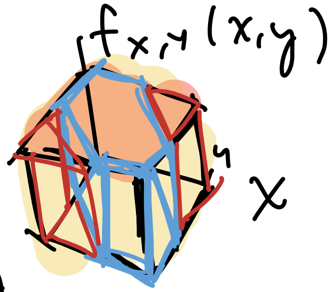
Recall: Finding the pdf of a transformation

- Let M be a transformation of X and Y: $M = g(X, Y)$
- When we have a transformation of X and Y, M, we need to follow the **CDF method** to find the pdf of M

We follow **CDF method**:

1. Start with the joint pdf for X and Y
   - aka $f_{X,Y}(x, y)$ ✓
2. Translate the domain of X and Y to M: find possible values of M ✓
3. Find the CDF of M
   - aka $F_M(m) = P(M \leq m) = P(g(X, Y) \leq m)$
4. Take the derivative of the CDF of M with respect to m to find the pdf of M
   - aka $f_M(m) = \dfrac{d}{dm} F_M(m)$

Example of a joint pdf with a transformation (1/2)



Example 4.1

Let X and Y have constant density on the square $0 \le X \le 4, 0 \le Y \le 4.$

1. Find $\mathbb{P}(|X - Y| < 2)$

$-2 < X - Y < 2$

constant density & volume must = 1

$4 \times 4 \times \frac{1}{16} = 1$

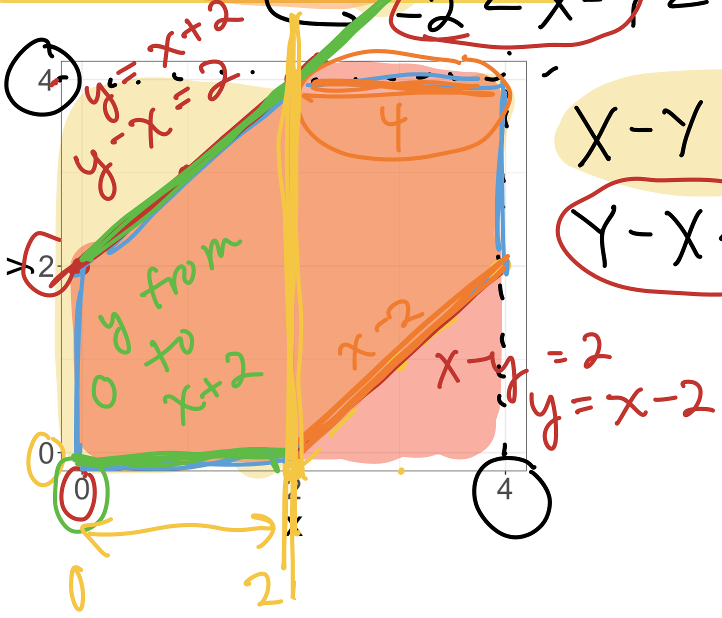
l w h

$h \rightarrow f_{x,y}(x,y)$

$P(|X-Y| < 2) =$

$X - Y < 2$

$Y - X < 2$



left side : $\displaystyle\int_0^2 \int_0^{x+2} \frac{1}{16}\, dy\, dx \; +$

right side $\displaystyle\int_2^4 \int_{x-2}^{4} \frac{1}{16}\, dy\, dx$

$$= \int_0^2 \frac{1}{16} y \Big|_{y=0}^{y=x+2} dx + \int_2^4 \frac{1}{16} y \Big|_{y=x-2}^{y=4} dx$$

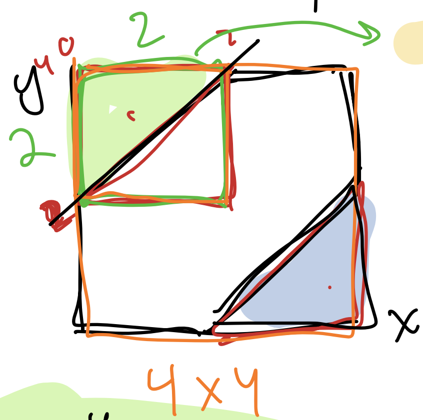
$$= \int_0^2 \frac{1}{16}(x+2)\, dx + \int_2^4 \left(\frac{3}{8} - \frac{1}{16}x\right) dx$$

$$= \frac{1}{32}x^2 + \frac{1}{8}x \Big|_0^2 + \frac{3}{8}x - \frac{1}{32}x^2 \Big|_2^4$$

$$= \frac{1}{32}(2)^2 + \frac{1}{8}2 - 0 + \frac{3}{8}4 - \frac{1}{32}(4)^2 - \frac{3}{8}(2) + \frac{1}{32}(2)^2$$

$$= \frac{3}{4}$$

ALT approach:



1/4 of domain

outside of region

$1 - \frac{1}{4} = \frac{3}{4}$ desired region

$$1 - \int_0^2 \int_{x+2}^4 \_\_\_ \, dy\,dx - \int_2^4 \int_{\cancel{x-2}\ 0}^{x-2} \_\_\_ \, dy\,dx$$

Example of a joint pdf with a transformation (1/2)

★ NEXT CLASS ★

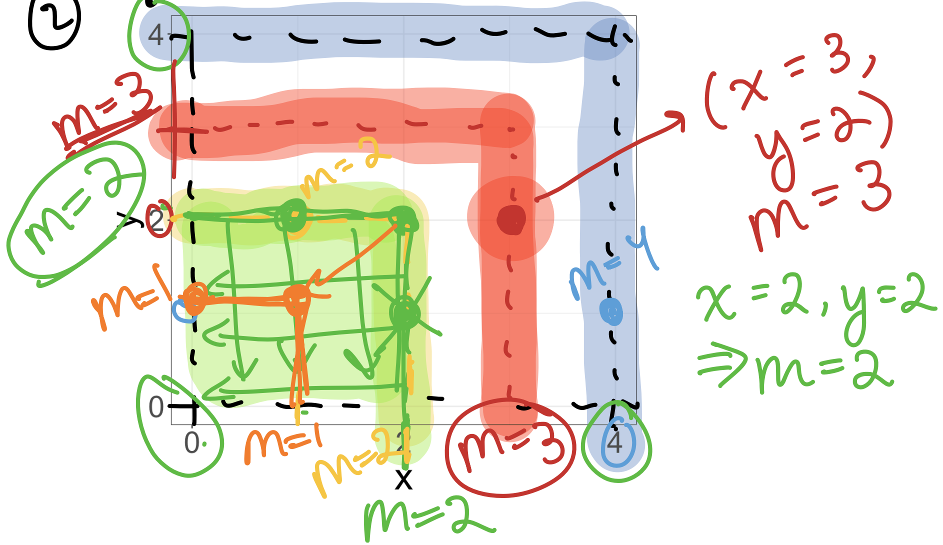
Example 4.2

Let X and Y have constant density on the square $0 \le X \le 4, 0 \le Y \le 4$.

2. Let $M = \max(X, Y)$. Find the pdf for M, that is $f_M(m)$

① $f_{X,Y}(x,y) = \frac{1}{16} \quad 0 \le x \le 4,\ 0 \le y \le 4$

② DOMAIN



③ $F_M(m) = P(M \le m) = P(\max(X,Y) \le m)$

$= P(X \le m, Y \le m)$

$= \int_0^m \int_0^m f_{X,Y}(x,y)\, dx\, dy$

$= \int_0^m \int_0^m \frac{1}{16}\, dx\, dy = \int_0^m \left[\frac{1}{16} x\right]_{x=0}^{x=m} dy$

$= \int_0^m \frac{1}{16} m\, dy = \left[\frac{1}{16} m y\right]_{y=0}^{y=m} = \frac{1}{16} m^2$

constant in y

④ $f_M(m) = \frac{d}{dm}\left(\frac{1}{16} m^2\right) = \frac{2m}{16} = \frac{1}{8} m$

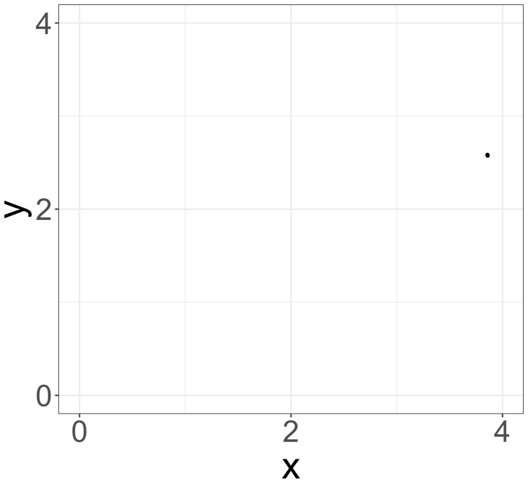
$f_M(m) = \frac{1}{8} m$ for $0 \le m \le 4$

Example of a joint pdf with a transformation (1/2)

Example 4.3

Let X and Y have constant density on the square $0 \le X \le 4, 0 \le Y \le 4$.

3. Let $Z = \min(X,Y)$. Find the pdf for Z, that is $f_Z(z)$.

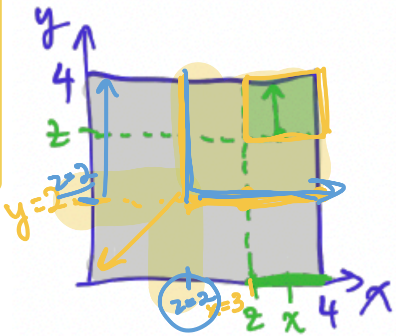


(3) *Let* $Z = \min(X,Y)$. *Find the pdf for* Z, *that is* $f_Z(z)$.

First find $F_Z(z)$.

$$F_Z(z) = P(Z \le z) = P(\min(X,Y) \le z)$$
$$= 1 - P(\min(X,Y) > z)$$
$$= 1 - P(X > z, Y > z)$$

Show these steps in your work!



$$= 1 - \int_z^4 \int_z^4 \frac{1}{16}\, dy\, dx$$

$$= 1 - \int_z^4 \frac{y}{16}\Big|_z^4 dx = 1 - \int_z^4 \frac{4-z}{16}\, dx$$

$$= 1 - \frac{(4-z)x}{16}\Big|_z^4 = 1 - \frac{(4-z)^2}{16} = \frac{z}{2} - \frac{z^2}{16}$$

$$F_Z(z) = \begin{cases} 0 & z < 0 \\ \frac{z}{2} - \frac{z^2}{16} & 0 \le z \le 4 \\ 1 & z > 4 \end{cases}$$

$$\Rightarrow f_Z(z) = F_Z'(z) = \frac{1}{2} - \frac{z}{8}$$

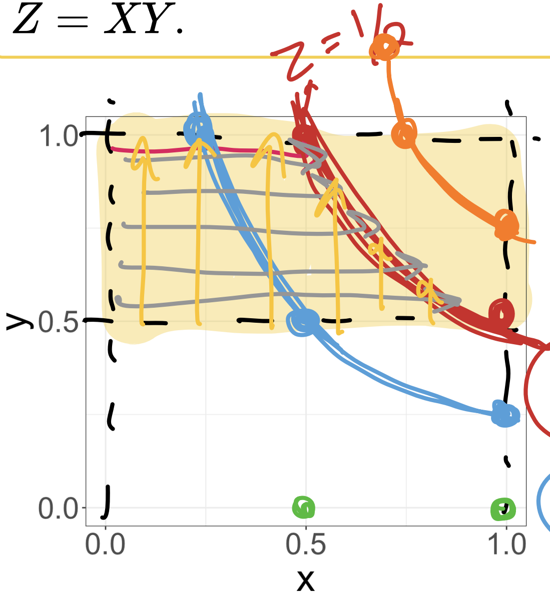
for $0 \le z \le 4$

Last example *for home*: more complicated transformation

Example 5

Let X and Y have joint density $f_{X,Y}(x,y) = \frac{8}{5}(x+y)$ in the region $0 < x < 1,\ \frac{1}{2} < y < 1$. Find the pdf of the RV Z, where $Z = XY$.

①



possible values of z:

$0 < x < 1,\ \frac{1}{2} < y < 1$

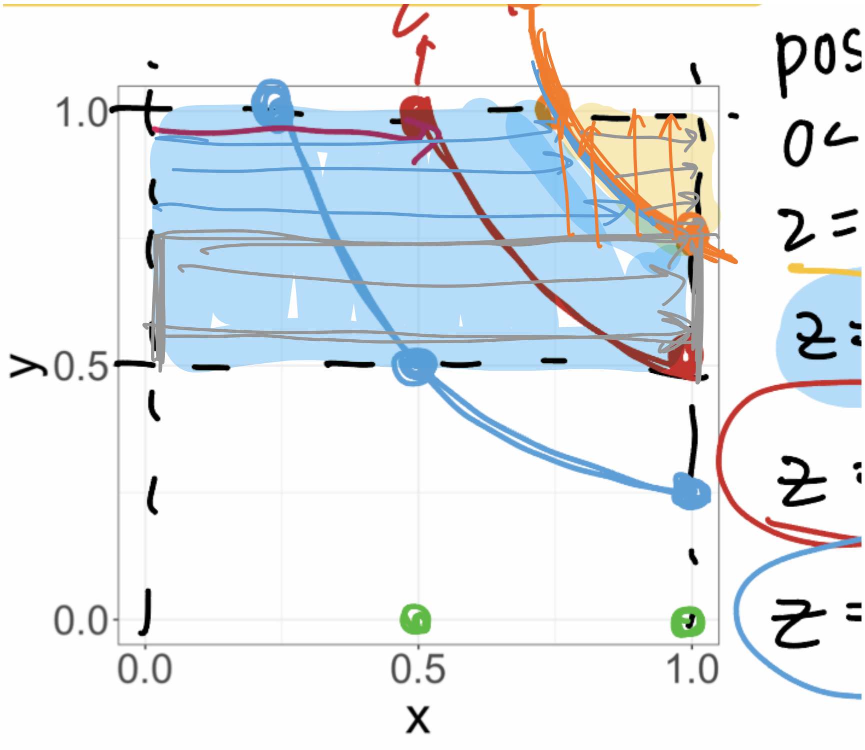
$z = xy \quad 0 < z < 1$

$z = 0$: $y = \frac{0}{x}$ $\quad x = 1 \Rightarrow y = 0$, $x = 1/2 \Rightarrow y = 0$

$z = \frac{1}{2}$: $y = \frac{(1/2)}{x} = \frac{1}{2x}$ $\quad x = 1 \Rightarrow y = 1/2$, $x = 1/2 \Rightarrow y = 1$

$z = 1/4$: $y = \frac{1}{4x}$ $\quad x = 1 \Rightarrow y = 1/4$, $x = 1/2 \Rightarrow y = 1/2$, $x = 1/4 \Rightarrow y = 1$

$z = \frac{3}{4}$: $y = \frac{3}{4x}$ $\quad x = 1 \Rightarrow y = \frac{3}{4}$, $x = 1/2 \Rightarrow y = \frac{3}{2}$, $x = 3/4 \Rightarrow y = 1$



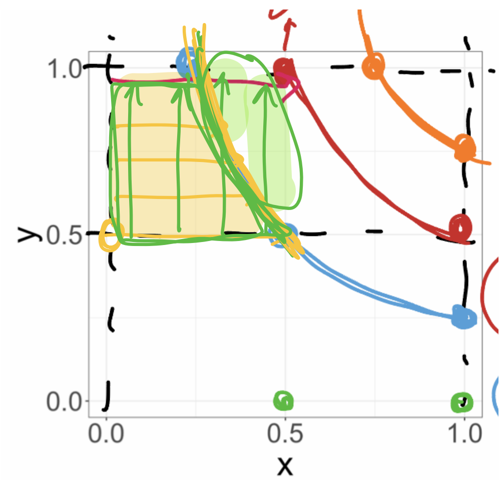
$z > 1/2$

$1 -$ yellow

$Z = XY$

$X = z/Y$

case 1: $0 < z < 1/2$



$$F_Z(z) = P(Z \le z) = P(XY \le z)$$

$$= \int_{1/2}^{1} \int_{0}^{z/y} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{1/2}^{1} \int_{0}^{z/y} \frac{8}{5}(x+y)\,dx\,dy = \ldots =$$

$$= \frac{4}{5}(z^2 + z)$$

case 2: $1/2 < z < 1$

Learning Objectives

1. Solve double integrals in our mini lesson!
2. Calculate probabilities for a pair of continuous random variables
3. Calculate a *joint and marginal* probability density function (pdf)
4. Calculate a *joint and marginal* cumulative distribution function (CDF) from a pdf

Double Integrals Mini Lesson (1/3)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

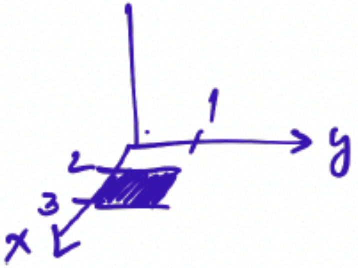
Mini Lesson Example 1

Solve the following integral:

$$\int_2^3 \int_0^1 xy\,dy\,dx$$

Example 25.1. *Solve the following integrals.*

(1) $\int_2^3 \left(\int_0^1 xy\,dy \right) dx$



$$= \int_2^3 \left(x \int_0^1 y\,dy \right) dx = \int_2^3 \left(x \left. \frac{y^2}{2} \right|_0^1 \right) dx$$

$y=$

$y=0$

$$= \int_2^3 x\left(\frac{1}{2} - 0 \right) dx = \int_2^3 \frac{x}{2}\,dx$$

$$= \left. \frac{x^2}{4} \right|_2^3 = \frac{1}{4}(9-4) = \boxed{\frac{5}{4}}$$

Double Integrals Mini Lesson (2/3)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Mini Lesson Example 2

Solve the following integral:

$$\int_2^3 \int_0^1 (x+y)dydx$$

(2) $\int_2^3 \int_0^1 (x+y)dydx$

$$= \int_2^3 \int_0^1 \underline{(x+y)}\, dy\, dx = \int_2^3 \left(xy + \frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1} dx$$

$$= \int_2^3 \left(x + \frac{1}{2} - 0\right) dx = \frac{x^2}{2} + \frac{x}{2}\Bigg|_2^3 = \frac{9}{2} + \frac{3}{2} - \left(\frac{4}{2} + \frac{2}{2}\right)$$

$$= \boxed{3}$$

Double Integrals Mini Lesson (3/3)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Mini Lesson Example 3

Solve the following integral:

$$\int_2^3 \int_0^1 e^{x+y} dydx$$

(3) $\int_2^3 \int_0^1 e^{x+y} dydx$

$$\int_2^3 \int_0^1 e^x e^y \, dydx = \int_2^3 e^x e^y \Big|_{y=0}^{y=1} dx = \int_2^3 e^x (e^1 - e^0) dx$$

$$= \int_2^3 (e-1) e^x dx = (e-1) e^x \Big|_2^3 = \boxed{(e-1)(e^3 - e^2)}$$