Lesson 12: Independence and Conditioning

Meike Niederhausen and Nicky Wakim

2025-11-03

Learning Objectives

1. Identify the formula for joint distributions for independent RVs and conditional distributions (PMFs/PDFs)
2. Find conditional pmf from a joint pmf and check if two RVs are independent.
3. Construct a joint distribution for two independent continuous RVs from their marginal distributions.
4. Calculate conditional probabilities and distributions for continuous RVs.

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Simulations
- Probability axioms
- Probability properties
- Counting
- Rules of probability

Probability for random variables

- Probability functions: pmfs/pdfs
- Cumulative distribution functions (CDFs)
- Important distributions

Two random variables

- Transformations
- Joint distributions
- Independence and conditioning

Expected values

- Means / expected values
- Linear combinations and conditioning
- Variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

Learning Objectives

1. Identify the formula for joint distributions for independent RVs and conditional distributions (PMFs/PDFs)
2. Find conditional pmf from a joint pmf and check if two RVs are independent.
3. Construct a joint distribution for two independent continuous RVs from their marginal distributions.
4. Calculate conditional probabilities and distributions for continuous RVs.

How do we represent conditional pmfs/pdfs?

For events:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

For discrete RVs:

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$p_{Y|X}(y|x) = P(Y = y|X = x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$$

if denominator is greater than 0 ($p_Y(y) > 0$ or $p_X(x) > 0$)

For continuous RVs:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

if denominator is greater than 0 ($f_Y(y) > 0$ or $f_X(x) > 0$)

How do we represent independent RVs in a joint pmf/pdf?

What do we know about independence for events?

For events: If $A \perp B$

$$P(A \cap B) = P(A)P(B)$$

$$P(A|B) = P(A)$$

For discrete RVs: If $X \perp Y$

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

$$p_{X|Y}(x|y) = p_X(x)$$

$$p_{Y|X}(y|x) = p_Y(y)$$

For continuous RVs: If $X \perp Y$

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

$$f_{X|Y}(x|y) = f_X(x)$$

$$f_{Y|X}(y|x) = f_Y(y)$$

Remember: our probability rules must hold for these!

For discrete RVs

For a valid joint pmf, we need:

- $0 \geq p_{X,Y}(x, y) \leq 1$ for all x, y
- $\sum_{\{all\ x\}} \sum_{\{all\ y\}} p_{X,Y}(x, y) = 1$

For a valid conditional pmf, we need:

- $0 \geq p_{X|Y}(x|y) \leq 1$ for all x, y
- $\sum_{\{all\ x\}} p_{X|Y}(x|y) = 1$

For continuous RVs

For a valid joint pdf, we need:

- $f_{X,Y}(x, y) \geq 0$ for all x, y
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$

For a valid conditional pdf, we need:

- $f_{X|Y}(x|y) \geq 0$ for all x and y
- $\int_{-\infty}^{\infty} f_{X|Y}(x|y) dx = 1$

Extra notes

- If $X_1, X_2, \ldots, X_n$ are **independent**

$$p_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{i=1}^{n} p_{X_i}(x_i)$$

$$f_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i)$$

$$F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n) = \prod_{i=1}^{n} P(X_i \le x_i) = \prod_{i=1}^{n} F_{X_i}(x_i)$$

- Don't forget, you can manipulate the conditional density to get the joint:

$$f_{X,Y}(x, y) = f_{X|Y}(x|y) f_Y(y)$$

Learning Objectives

1. Identify the formula for joint distributions for independent RVs and conditional distributions (PMFs/PDFs)
2. Find conditional pmf from a joint pmf and check if two RVs are independent.
3. Construct a joint distribution for two independent continuous RVs from their marginal distributions.
4. Calculate conditional probabilities and distributions for continuous RVs.

Last class: joint distribution for two discrete random variables (1/2)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

7. Find $p_{X|Y}(x|y)$.

		Y			
		1	2	3	
	1				
X	2				
	3				

Last class: joint distribution for two discrete random variables (2/2)

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

8. Are X and Y independent? Why or why not?

		Y			
		1	2	3	
	1				
X	2				
	3				

Remark:

- To show that X and Y are *not* independent, we just need to find one counter example
- However, to show that they *are* independent, we need to verify this for all possible pairs of x and y

Learning Objectives

1. Identify the formula for joint distributions for independent RVs and conditional distributions (PMFs/PDFs)
2. Find conditional pmf from a joint pmf and check if two RVs are independent.
3. Construct a joint distribution for two independent continuous RVs from their marginal distributions.
4. Calculate conditional probabilities and distributions for continuous RVs.

Constructing a joint pdf from two independent, continuous RVs

Example 1.1

Let X and Y be independent r.v.'s with $f_X(x) = \frac{1}{2}$, for $0 \leq x \leq 2$ and $f_Y(y) = 3y^2$, for $0 \leq y \leq 1$.

1. Find $f_{X,Y}(x, y)$

Probability from joint pdf from two independent, continuous RVs

Example 1.2

Let X and Y be independent r.v.'s with $f_X(x) = \frac{1}{2}$, for $0 \leq x \leq 2$ and $f_Y(y) = 3y^2$, for $0 \leq y \leq 1$.

2. Find $\mathbb{P}(0 \leq X \leq 1, 0 \leq Y \leq \frac{1}{2})$

Showing independence from joint pdf

Example 2.1

Let $f_{X,Y}(x,y) = 18x^2y^5$, for $0 \le x \le 1,\ 0 \le y \le 1.$

1. Are X and Y independent?

Finding CDF from two independent RVs

Example 2.2

Let $f_{X,Y}(x,y) = 18x^2y^5$, for $0 \leq x \leq 1,\ 0 \leq y \leq 1.$

2. Find $F_{X,Y}(x,y)$.

Showing independence from joint pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 3

Let $f_{X,Y}(x,y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$. Are X and Y independent?

Final statement on independence

1. If $f_{X,Y}(x,y) = g(x)h(y)$, where $g(x)$ and $h(y)$ are pdf's, then X and Y are independent.
   - The domain of the joint pdf needs to be independent as well!!

2. If $F_{X,Y}(x,y) = G(x)H(y)$, where $G(x)$ and $H(y)$ are cdf's, then X and Y are independent.
   - The domain of the joint CDF needs to be independent as well!!

- Make sure that:
  - X domain does NOT depend on Y
  - Y domain does NOT depend on X

Learning Objectives

1. Identify the formula for joint distributions for independent RVs and conditional distributions (PMFs/PDFs)
2. Find conditional pmf from a joint pmf and check if two RVs are independent.
3. Construct a joint distribution for two independent continuous RVs from their marginal distributions.
4. Calculate conditional probabilities and distributions for continuous RVs.

Example starting from a joint pdf: first try!

Example 1.1

Let $f_{X,Y}(x,y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

1. Find $\mathbb{P}(2 < X < 10 | Y = 4)$

Example starting from a joint pdf: second try! (1/2)

Example 1.1

Let $f_{X,Y}(x, y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

1. Find $\mathbb{P}(2 < X < 10 | Y = 4)$

Example starting from a joint pdf: second try! (2/2)

Example starting from a joint pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 1.2

Let $f_{X,Y}(x, y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

2. Find $\mathbb{P}(X > 20|Y = 5)$

Finding probability with conditional domain and pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 2

Randomly choose a point X from the interval $[0, 1]$, and given $X = x$, randomly choose a point Y from $[0, x]$. Find $\mathbb{P}(0 < Y < \frac{1}{4})$.