Lesson 13: Expected Values

Meike Niederhausen and Nicky Wakim

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Learning Objectives

1. Define the expected value for discrete and continuous RVs
2. Calculate the expected value (mean) of a single discrete RV
3. Calculate the expected value (mean) of a single continuous RV
4. Calculate expected value for the sum of discrete or continuous RVs
5. Calculate expected value for joint densities (continuous RVs)

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Simulations
- Probability axioms
- Probability properties
- Counting
- Rules of probability

Probability for random variables

- Probability functions: pmfs/pdfs
- Cumulative distribution functions (CDFs)
- Important distributions

Two random variables

- Transformations
- Joint distributions
- Independence and conditioning

Expected values

- Means / expected values
- Linear combinations and conditioning
- Variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

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Our good and fair friend, the 6-sided die

Example 1

Suppose you roll a fair 6-sided die. What value do you expect to get?

What is an expected value?

Definition: Expected value

The **expected value** of a **discrete RV** X that takes on values $x_1, x_2, \ldots, x_n$ is

$$\mathbb{E}[X] = \sum_{i=1}^{n} x_i p_X(x_i)$$

where n can be ∞

Definition: Expected value

The **expected value** of a **continuous RV** X is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

where we adjust the integrand based on the bounds of X

- Expected values are not necessarily an actual outcome
  - In previous example, we cannot roll a 3.5
  - It could be that our expected value is not in the sample space ($E(X) \notin S$)

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Our good and not-so-fair friend, the 6-sided die

Example 2

Suppose the die is 6-sided, but not fair. And the probabilities of each side is distributed as:

x	$p_X(x)$
1	0.10
2	0.05
3	0.02
4	0.30
5	0.50
6	0.03

What value do you expect to get on a roll?

Expected value of a Bernoulli distribution

Example 3

Suppose

$$X = \begin{cases} 1 & \text{with probability } p \quad \text{(success)} \\ 0 & \text{with probability } 1 - p \quad \text{(failure)} \end{cases}$$

Find the expected value of X.

Let's slightly change our random variable

Example 5

Suppose

$$X = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p \end{cases}$$

Find the expected value of X.

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Expected Value of the Uniform Distribution

Example 1

Let $f_X(x) = \frac{1}{b-a}$, for $a \leq x \leq b$. Find $\mathbb{E}[X]$.

Expected Value of the Exponential Distribution

Example 2

Let $f_X(x) = \lambda e^{-\lambda x}$, for $x > 0$ and $\lambda > 0$. Find $\mathbb{E}[X]$.

Integrating by Parts

$$\int_a^b u dv = uv\Big|_a^b - \int_a^b v du$$

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Revisiting our two card draw

Example 1

Suppose you draw 2 cards from a standard deck of cards *with* replacement. Let X be the number of hearts you draw. Find $\mathbb{E}[X]$.

Recall Binomial RV with $n = 2$:

$$p_X(x) = \binom{2}{x} p^x (1-p)^{2-x} \text{ for } x = 0, 1, 2$$

Sums of Random Variables

Theorem: Sum of random variables

For RVs (discrete or continuous) X_i and constants $a_i, i = 1, 2, \ldots, n$,

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i].$$

Remark: The theorem holds for infinitely RV's X_i as well.

- For two RVs, X and Y:
  - We can say $E[X+Y] = E[X] + E[Y]$
  - … and constant numbers a and b, we can also say $E[aX + bY] = aE[X] + bE[Y]$
  - We can also also say $E[X - Y] = E[X] - E[Y]$, since $b = -1$

Corollaries from Theorem

Function with two constants

For a RV X, and constants a and b,

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b.$$

Expected value of sum of identically distributed RVs

If $X_i, i = 1, 2, \ldots, n$, are identically distributed RV's, then

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = n\mathbb{E}[X_1].$$

Cost of hotel rooms

Example 4

A tour group is planning a visit to the city of Minneapolis and needs to book 30 hotel rooms. The average price of a room is \$200. In addition, there is a 10% tourism tax for each room. What is the expected cost for the 30 hotel rooms?

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Expected value of one RV from joint pdf

If you have a joint distribution $f_{X,Y}(x, y)$ and want to calculate $\mathbb{E}[X]$, you have two options:

1. Find $f_X(x)$ and use it to calculate $\mathbb{E}[X]$.
2. Calculate $\mathbb{E}[X]$ using the joint density:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x, y) dy dx.$$

You can do the same for $\mathbb{E}[Y]$!

Option 1: Find marginal first

Example 3

Let $f_{X,Y}(x,y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$. Find $\mathbb{E}[X]$.

Do this one at home by finding $f_X(x)$ then $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$. See if you get the same result as next page's answer!

Option 2: Expected value from a joint distribution

Example 1

Let $f_{X,Y}(x,y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$. Find $\mathbb{E}[X]$.

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