Lesson 13: Expected Values

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2025-11-10

Learning Objectives

1. Define the expected value for discrete and continuous RVs
2. Calculate the expected value (mean) of a single discrete RV
3. Calculate the expected value (mean) of a single continuous RV
4. Calculate expected value for the sum of discrete or continuous RVs
5. Calculate expected value for joint densities (continuous RVs)

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Simulations
- Probability axioms
- Probability properties
- Counting
- Rules of probability

Probability for random variables

- Probability functions: pmfs/pdfs
- Cumulative distribution functions (CDFs)
- Important distributions

Two random variables

- Transformations
- Joint distributions
- Independence and conditioning

Expected values

- Means / expected values
- Linear combinations and conditioning
- Variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

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Our good and fair friend, the 6-sided die

Example 1

Suppose you roll a fair 6-sided die. What value do you expect to get?

$$\text{avg} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6}$$

$$= 3.5$$

$$\text{weighted avg} = \underbrace{\left(\frac{1}{6}\right)}_{P(X=1)}1 + \underbrace{\left(\frac{1}{6}\right)}_{P(X=2)}2 + \left(\frac{1}{6}\right)3 + \left(\frac{1}{6}\right)4 + \left(\frac{1}{6}\right)5 + \left(\frac{1}{6}\right)6$$

$$= 3.5$$

What is an expected value?

Definition: Expected value

The **expected value** of a **discrete RV** X that takes on values $x_1, x_2, \ldots, x_n$ is

$$\mathbb{E}[X] = \sum_{i=1}^{n} x_i p_X(x_i)$$

where n can be ∞

Definition: Expected value

The **expected value** of a **continuous RV** X is

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

where we adjust the integrand based on the bounds of X

- Expected values are not necessarily an actual outcome
  - In previous example, we cannot roll a 3.5
  - It could be that our expected value is not in the sample space ($E(X) \notin S$)

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Our good and not-so-fair friend, the 6-sided die

Example 2

Suppose the die is 6-sided, but not fair. And the probabilities of each side is distributed as:

x	$p_X(x)$
1	0.10
2	0.05
3	0.02
4	0.30
5	0.50
6	0.03

weighted by

What value do you expect to get on a roll?

$$E(X) = \sum_{i=1}^{6} x_i P(X = x_i)$$

$$= 1(0.1) + 2(0.05) + 3(0.02) + 4(0.3) + 5(0.5) + 6(0.03)$$

$$= \underline{4.14}$$

★ do NOT round E(X) to nearest whole number

Expected value of a Bernoulli distribution

Example 3

Suppose

$$X = \begin{cases} 1 & \text{with probability } p \quad (\text{success}) \\ 0 & \text{with probability } 1-p \quad (\text{failure}) \end{cases}$$

Find the expected value of X.

$$E(X) = \sum_{i=1}^{2} x_i P(X = x_i) = 0(1-p) + 1p$$

$$= p$$

Let's slightly change our random variable

Example 5

Suppose

$$X = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1 - p \end{cases}$$

Find the expected value of X.

$$E(X) = \sum_{i=1}^{2} x_i P(X = x_i) = (1)p + (-1)(1-p)$$

$$= p - (1-p) = 2p - 1$$

$p = \frac{1}{2}$: $E(X) = 0$

$p > \frac{1}{2}$: $E(X) > 0$

$p < \frac{1}{2}$: $E(X) < 0$

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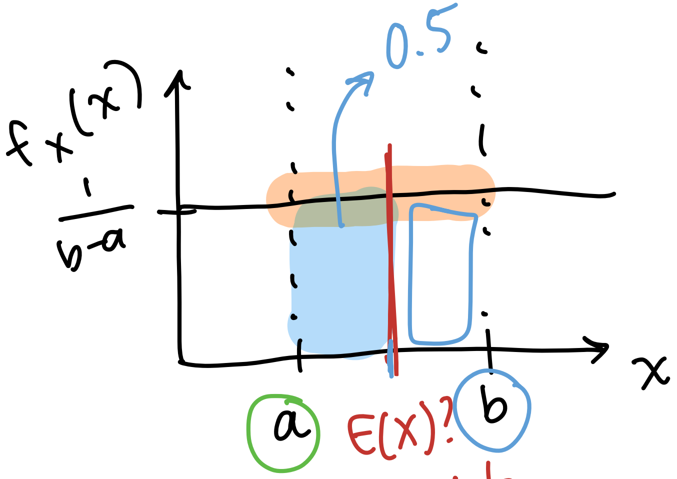
Expected Value of the Uniform Distribution

Example 2

Let $f_X(x) = \frac{1}{b-a}$, for $a \le x \le b$. Find $\mathbb{E}[X]$.



$E(X)$? $= \frac{a+b}{2}$

$$E(X) = \int_a^b x f_X(x)\, dx$$

$$= \int_a^b x \left(\frac{1}{b-a}\right) dx$$

$$= \left(\frac{1}{b-a}\right)\left(\frac{1}{2}\right) x^2 \Big|_{x=a}^{x=b}$$

$$= \frac{1}{2(b-a)}\left[b^2 - a^2\right]$$

$$x^2 - y^2 = (x+y)(x-y)$$

$$= \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$= \frac{b+a}{2}$$

Expected Value of the Exponential Distribution

Example 3

Let $f_X(x) = \lambda e^{-\lambda x}$ for $x > 0$ and $\lambda > 0$. Find $\mathbb{E}[X]$.

Integrating by Parts

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$$\lim_{x\to\infty} x e^{-x} = 0$$

$$E(X) = \int_0^\infty x f_X(x)\,dx$$

$$= \int_0^\infty x\left(\lambda e^{-\lambda x}\right)dx$$

$$= \left[\cancel{\lambda} x\left(-\frac{1}{\cancel{\lambda}} e^{-\lambda x}\right)\right]_{x=0}^{x=\infty} + \int_0^\infty \frac{1}{\cancel{\lambda}} e^{-\lambda x}\,\cancel{\lambda}\,dx$$

Integrate parts

$u = \lambda x \qquad du = \lambda dx$

$dv = e^{-\lambda x}dx$

$v = \frac{-1}{\lambda} e^{-\lambda x}$

$$\frac{d}{dx}\left(\frac{-1}{\lambda} e^{-\lambda x}\right) = -\frac{1}{\cancel{\lambda}}\left(-\cancel{\lambda}\, e^{-\lambda x}\right)$$

$$= 0 - 0 + \int_0^\infty e^{-\lambda x}\,dx = \frac{-1}{\lambda} e^{-\lambda x}\Big|_{x=0}^{x=\infty}$$

$$= \cancel{\frac{-1}{\lambda} e^{-\infty}}^{\,0} - \left(\frac{-1}{\lambda}\right)\cancel{e^{-\lambda(0)}}^{\,1} = \frac{1}{\lambda}$$

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Revisiting our two card draw

Example 1

Suppose you draw 2 cards from a standard deck of cards with replacement. Let X be the number of hearts you draw. Find $\mathbb{E}[X]$.

Recall Binomial RV with $n = 2$:

$$p_X(x) = \binom{2}{x} p^x (1-p)^{2-x} \text{ for } x = 0, 1, 2$$

$$P(\heartsuit) = p = \frac{13}{52} = \frac{1}{4} = 0.25$$

$$E(X) = \sum_{i=1}^{3} x_i P(X = x_i)$$

$$= 0 \cdot P(X=0) + 1\, P(X=1) + 2\, P(X=2)$$

$$= 0 \cdot \binom{2}{0} 0.25^0\, 0.75^2 + 1 \binom{2}{1} 0.25^1\, 0.75^1 + 2 \binom{2}{2} 0.25^2\, 0.75^0$$

★ Binomial(n, p) is the sum of n Bernoulli(p) $Y \sim$

$\hookrightarrow E(Y) = p$

$$X = \sum_{i=1}^{2} Y_i$$

$$E(X) = E\left[\sum_{i=1}^{2} Y_i\right] = E(Y_1) + E(Y_2) = p + p = 2p = np$$

Sums of Random Variables

Theorem: Sum of random variables

For RVs (discrete or continuous) X_i and constants $a_i, i = 1, 2, \ldots, n$,

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i].$$

Remark: The theorem holds for infinitely RV's X_i as well.

- For two RVs, X and Y:
  - We can say $E[X+Y] = E[X] + E[Y]$
  - ... and constant numbers a and b, we can also say $E[aX + bY] = aE[X] + bE[Y]$
  - We can also also say $E[X - Y] = E[X] - E[Y]$, since $b = -1$

Corollaries from Theorem

Function with two constants

For a RV X, and constants a and b,

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b.$$

Expected value of sum of identically distributed RVs

If X_i, $i = 1, 2, \ldots, n$, are $\underbrace{\text{identically distributed}}_{\text{same dist, same parameters}}$ RV's, then

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = n\mathbb{E}[X_1].$$

$$\sum_{i=1}^{n} E(X_i) = n E(X_n)$$

$$E(X_1) = E(X_2) = \ldots = E(X_n)$$

Cost of hotel rooms

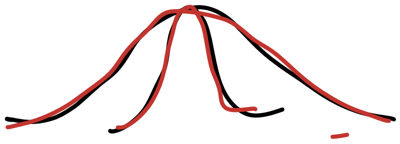
let: C_i = cost of room i

T = total cost of 30 rooms

Example 4 ★NOT necessarily identical

A tour group is planning a visit to the city of Minneapolis and needs to book 30 hotel rooms. The average price of a room is \$200. In addition, there is a 10% tourism tax for each room. What is the expected cost for the 30 hotel rooms?

$E(C_i) = 200$



$$T = \sum_{i=1}^{30} 1.1 C_i$$

$$E(T) = E\left[\sum_{i=1}^{30} 1.1 C_i\right] = \sum_{i=1}^{30} E[1.1 C_i] = \sum_{i=1}^{30} 1.1 E(C_i)$$

$\sum a X_i$
$= a \sum X_i$

$$= 1.1 \sum_{i=1}^{30} E(C_i) = 1.1 \sum_{i=1}^{30} 200$$

$$= 1.1\,(\underset{1}{200} + 200 + \dots + \underset{30}{200}) = 1.1 \cdot 30 \cdot 200$$

$$= \$6{,}600$$

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Expected value of one RV from joint pdf

If you have a joint distribution $f_{X,Y}(x,y)$ and want to calculate $\mathbb{E}[X]$, you have two options:

1. Find $f_X(x)$ and use it to calculate $\mathbb{E}[X]$.
2. Calculate $\mathbb{E}[X]$ using the joint density:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{X,Y}(x,y)\,dy\,dx.$$

You can do the same for $\mathbb{E}[Y]$!

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\right] dx$$

Option 1: Find marginal first

Example 3

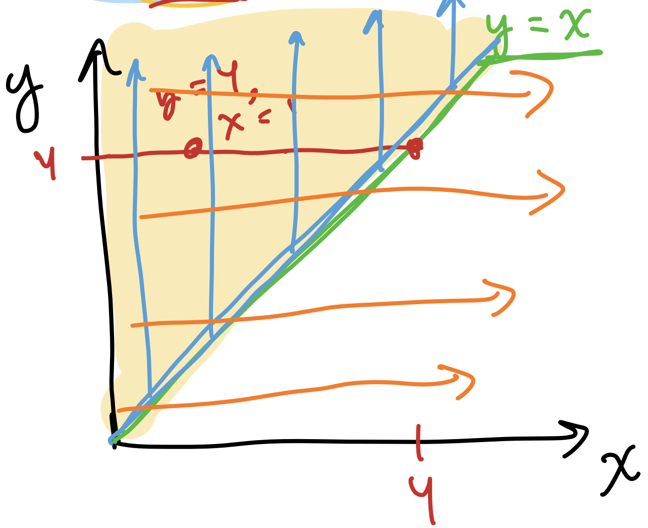
Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$. Find $\mathbb{E}[X]$.

Do this one at home by finding $f_X(x)$ then $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$. See if you get the same result as next page's answer!

Option 2: Expected value from a joint distribution

Example 1

Let $f_{X,Y}(x,y) = 2e^{-(x+y)}$, for $0 \le x \le y$. Find $\mathbb{E}[X]$.



$0 \le x$

$$E(x) = \int_0^\infty \int_x^\infty x f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^\infty \int_x^\infty x\,2e^{-(x+y)}\,dy\,dx$$

$$= \int_0^\infty \int_x^\infty x\,2e^{-x} e^{-y}\,dy\,dx$$

$$= \int_0^\infty x\,2e^{-x} \int_x^\infty e^{-y}\,dy\,dx$$

$$= \int_0^\infty x\,2e^{-x} \Big[-e^{-y}\Big]_{y=x}^{y=\infty}\,dx$$

$$= \int_0^\infty x\,2e^{-2x}\,dx = \ldots = \frac{1}{2}$$

int by parts

OR

exponential dist:

$f_X(x) = \lambda e^{-\lambda x}$

$E(X) = \frac{1}{\lambda}$

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HW 7 #2 pt a

$$\bar{X} = \frac{\sum X_i}{n}$$

$$E(\bar{X}) = E\left(\frac{\sum X_i}{n}\right)$$

n is constant

$$= \frac{1}{n} E\left(\sum X_i\right)$$

sum id. dist RVs

$$= \frac{1}{\cancel{n}}\left(\cancel{n}\, E(X_1)\right)$$

$$= E(X_1)$$