Lesson 14: Variance

Meike Niederhausen and Nicky Wakim

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Learning Objectives

1. Define and calculate the expected value for a function of discrete and continuous RVs
2. Define and calculate variance for a single random variable
3. Define and calculate variance for multiple random variables

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Simulations
- Probability axioms
- Probability properties
- Counting
- Rules of probability

Probability for random variables

- Probability functions: pmfs/pdfs
- Cumulative distribution functions (CDFs)
- Important distributions

Two random variables

- Transformations
- Joint distributions
- Independence and conditioning

Expected values

- Means / expected values
- Linear combinations and conditioning
- Variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

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Let's start building the variance through expected values of functions

Example 1

Let g be a function and let $g(x) = ax + b$, for real-valued constants a and b. What is $\mathbb{E}[g(X)]$?

What is the expected value of a function?

Expected value of function of discrete RV

For any function g and discrete RV X, the expected value of $g(X)$ is

$$\mathbb{E}[g(X)] = \sum_{\{all\ x\}} g(x)p_X(x)$$

Expected value of function of continuous RV

For any function g and continuous RV X, the expected value of $g(X)$ is

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x)dx$$

- For example, if we have $g(x) = x^2$, then

$$\mathbb{E}[X^2)] = \sum_{\{all\ x\}} x^2 p_X(x) \neq \left(\sum_{\{all\ x\}} x p_X(x))\right)^2 = (\mathbb{E}[X])^2$$

Let's revisit the card example (1/2)

Example 2

Suppose you draw 2 cards from a standard deck of cards *with* replacement. Let X be the number of hearts you draw.

1. Find $\mathbb{E}[X^2]$.

Recall Binomial RV with $n = 2$:

$$p_X(x) = \binom{2}{x} p^x (1-p)^{2-x} \text{ for } x = 0, 1, 2$$

Let's revisit the card example (2/2)

Example 2

Suppose you draw 2 cards from a standard deck of cards *with* replacement. Let X be the number of hearts you draw.

2. Find $\mathbb{E}\left[\left(X - \frac{1}{2}\right)^2\right]$.

Recall Binomial RV with $n = 2$:

$$p_X(x) = \binom{2}{x} p^x (1-p)^{2-x} \text{ for } x = 0, 1, 2$$

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Variance of a RV

Definition: Variance of RV

The variance of a RV X, with (finite) expected value $\mu_X = \mathbb{E}[X]$ is

$$\sigma_X^2 = Var(X) = \mathbb{E}[(X - \mu_X)^2] = \mathbb{E}[(X - \mathbb{E}[X])^2]$$

Definition: Standard deviation of RV

The standard deviation of a RV X is

$$\sigma_X = SD(X) = \sqrt{\sigma_X^2} = \sqrt{Var(X)}.$$

Variance of discrete and continuous RVs

How do we calculate the variance of a discrete RV?

For discrete RVs:

$$
\begin{aligned}
Var(X) &= \\
&= \sum_{\{all\ x\}} (x - \mu_x)^2 p_X(x) \\
&= \mathbb{E}[(X - \mu_X)^2] \\
&= \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2
\end{aligned}
$$

How do we calculate the variance of a continuous RV?

For continuous RVs:

$$
\begin{aligned}
Var(X) &= \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) dx \\
&= \mathbb{E}[(X - \mu_X)^2] \\
&= \mathbb{E}[(X - \mathbb{E}[X])^2] \\
&= \mathbb{E}[X^2] - (\mathbb{E}[X])^2
\end{aligned}
$$

Let's calculate the variance and prove it! (1/2)

Lemma 6: "Computation formula" for Variance

The variance of a RV X, can be computed as

$$\begin{aligned} \sigma_X^2 &= Var(X) \\ &= \mathbb{E}[X^2] - \mu_X^2 \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \end{aligned}$$

Let’s calculate the variance and prove it! (2/2)

Variance of an Uniform distribution

Example 2

Let $f_X(x) = \frac{1}{b-a}$, for $a \leq x \leq b$. Find $Var[X]$.

Variance of exponential distribution

In the homework:

Example 3

Let $f_X(x) = \lambda e^{-\lambda x}$, for $x > 0$ and $\lambda > 0$. Find $Var[X]$.

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Variance of a function with a single RV

Lemma 7

For a RV X and constants a and b,

$$Var(aX + b) = a^2 Var(X).$$

Proof will be exercise in homework. It's fun! In a mathy kinda way.

Important results for *independent* RVs

Theorem 8

For independent RV's X and Y, and functions g and h,

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)].$$

Corollary 1

For independent RV's X and Y,

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y].$$

Variance of sum of independent discrete RVs

Theorem 9: Variance of sum of independent discrete RV's

For independent discrete RV's X_i and constants $a_i, i = 1, 2, \ldots, n$,

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i).$$

Simpler version:

$$Var(a_1 X + a_2 Y) = Var(a_1 X) + Var(a_2 Y) = a_1^2 Var(X) + a_2^2 Var(Y)$$

Corollaries

Corollary 2

For independent discrete RV's $X_i, i = 1, 2, \ldots, n$,

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i).$$

Corollary 3

For independent identically distributed (i.i.d.) discrete RV's $X_i, i = 1, 2, \ldots, n$,

$$Var\left(\sum_{i=1}^{n} X_i\right) = nVar(X_1).$$

Ghost problem: *with replacement*

Example 3.2

The ghost is trick-or-treating at a different house now. In this case it is known that the bag of candy has 10 chocolates, 20 lollipops, and 30 laffy taffies. The ghost grabs a handful of five pieces of candy. What is the variance for the number of chocolates the ghost takes? Let's solve this for the cases ***with*** replacement.

Recall probability with replacement:

$$p_X(x) = \binom{n}{k} p^k (1-p)^{n-k}$$

Back to our hotel example from Lesson 13

Example 4

A tour group is planning a visit to the city of Minneapolis and needs to book 30 hotel rooms. The average price of a room is \$200 **with standard deviation \$10**. In addition, there is a 10% tourism tax for each room. What is the **standard deviation** of the cost for the 30 hotel rooms? Assume rooms are independent.

Problem to do at home if we don't have enough time.

Find the mean and sd from word problem (1/2)

Example 4

A machine manufactures cubes with a side length that varies uniformly from 1 to 2 inches. Assume the sides of the base and height are equal. The cost to make a cube is 10 ¢ per cubic inch, and 5 ¢ cents for the general cost per cube. Find the mean and standard deviation of the cost to make 10 cubes.

Find the mean and sd from word problem (1/2)