

Lesson 15: Some Important Discrete RVs

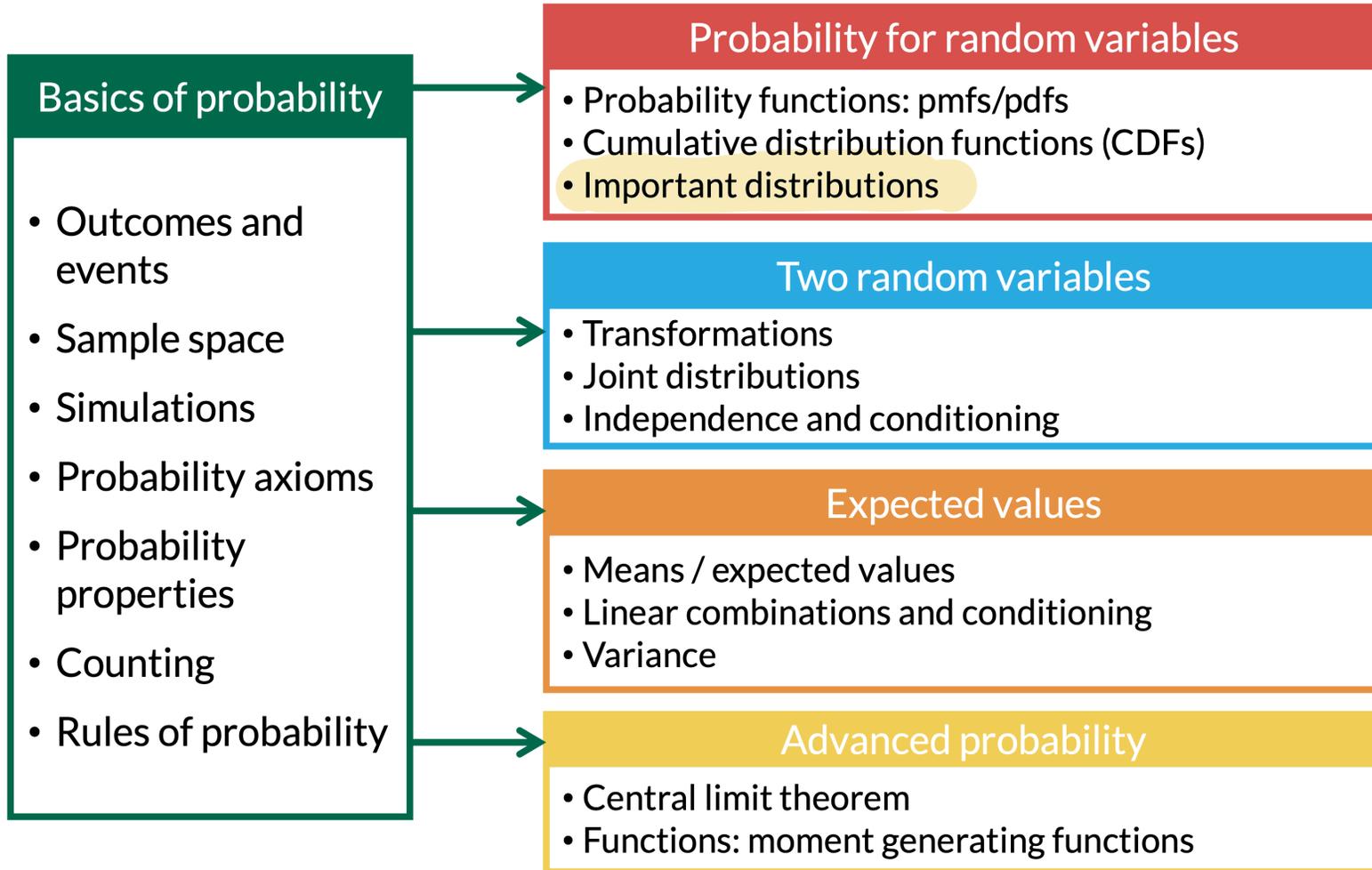
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Learning Objectives

1. Distinguish between Bernoulli, Binomial, Geometric, Hypergeometric, Discrete Uniform, Negative Binomial, and Poisson distributions when reading a world problem.
2. Identify the variable and the parameters in a world problem, and state what the variable and parameters mean.
3. Use the formulas for the pmf/CDF, expected value, and variance to answer questions and find probabilities.

Where are we?



Bernoulli RVs

Properties of Bernoulli RVs

- Scenario: One trial, with outcome success or failure
- Shorthand: $X \sim \text{Bernoulli}(p)$

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

pmf

$$p_X(x) = P(X = x) = p^x(1 - p)^{1-x} \text{ for } x = 0, 1$$

$$E(X) = p$$

$$\text{Var}(X) = pq = p(1 - p)$$

Bernoulli Example 1

Example 1

- We roll a fair 6-sided die.
- We get \$1 if we roll a 5, and nothing otherwise.
- Let X be how much money we get.
- Find the mean and variance of X .

$X = \$$ we get

$$X = \begin{cases} 1 & \text{roll a 5} \\ 0 & \text{roll anything else} \end{cases} \quad p = \frac{1}{6} \quad 1-p = \frac{5}{6}$$

$$X \sim \text{Bernoulli}(p = \frac{1}{6})$$

$$E(X) = p = \frac{1}{6}$$

$$\text{Var}(X) = p(1-p) = \frac{1}{6} \left(\frac{5}{6} \right) = \frac{5}{36}$$

Binomial RVs

Properties of Binomial RVs

- **Scenario:** There are n independent trials, each resulting in a success or failure, with constant probability, p , in each trial. We are counting the number of successes (or failures).
- Shorthand: $X \sim \text{Binomial}(n, p)$

X = Number of successes of n independent trials

$$p_X(x) = P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \text{ for } x = 0, 1, 2, \dots, n$$

$$E(X) = np$$

$$\text{Var}(X) = npq = np(1 - p)$$

[Binomial is the sum of n independent Bernoulli's]

Our beloved fair-sided die

Example 2

- Suppose we roll a fair 6-sided die 50 times.
- We get \$1 every time we roll a 5, and nothing otherwise.
- Let Y be how much money we get on the 50 rolls.
- Find the mean and variance of Y .

let $Y = \$$ after 50 rolls

$$n = 50 \quad p = \frac{1}{6}$$

$$Y = \sum_{i=1}^{50} X_i$$

$X_i \stackrel{iid}{\sim} \text{Bern}\left(\frac{1}{6}\right)$

iid = independent
& identically
distributed

$$E(Y) = np = 50 \left(\frac{1}{6}\right) = \frac{50}{6}$$

$$\text{Var}(Y) = np(1-p) = 50 \left(\frac{1}{6}\right) \left(\frac{5}{6}\right) = \frac{250}{36}$$

Geometric RVs

Geometric RVs

- **Scenario:** There are repeated independent trials, each resulting in a success or failure, with constant probability of success for each trial. We are counting the number of trials until the first success.
- Shorthand: $X \sim \text{Geo}(p)$ or $X \sim \text{Geometric}(p)$ or $X \sim G(p)$ Coin toss until H.

(X) = Number of trials needed for first success (count x includes the success)

(X) = Number of failures before first success (count x does not include the success)

H or TH or TTH
or TTTH

$$P_x(x) = P(X=x) = (1-p)^{x-1} p$$

for $x = 1, 2, 3, 4, \dots$

$$P_x(x) = P(X=x) = (1-p)^x p$$

for $x = 0, 1, 2, 3, 4, \dots$

$$F_x(x) = P(X \leq x) = 1 - (1-p)^x$$

for $x = 1, 2, 3, \dots$

$$F_x(x) = P(X \leq x) = 1 - (1-p)^{x+1}$$

for $x = 0, 1, 2, 3, 4, \dots$

$$E(X) = \frac{1}{p}$$
$$Var(X) = \frac{1-p}{p^2}$$

$$E(X) = \frac{1-p}{p}$$
$$Var(X) = \frac{1-p}{p^2}$$

Bullseye (1/4)

Example 3

We throw darts at a dartboard until we hit the bullseye. Assume throws are independent and the probability of hitting the bullseye is 0.01 for each throw.

1. What is the pmf for the number of throws needed to hit the bullseye? *includes success*
2. What are the mean and variance for the number of throws needed to hit the bullseye?
3. Find the probability that our first bullseye:
 - is on one of the first fifty tries
 - is after the 50th try, given that it did not happen on the first 20 tries

Bullseye (2/4)

Example 3

We throw darts at a dartboard until we hit the bullseye. Assume throws are independent and the probability of hitting the bullseye is 0.01 for each throw.

1. What is the pmf for the number of throws needed to hit the bullseye?

$X = \#$ trials w/ 1 success

series:

- H
- M H
- M M H
- M M M H
- \vdots

$$P(X=x) = (1-p)^{x-1} p$$

$$P_x(x) = 0.99^{x-1} 0.01$$

for $x=1, 2, 3, \dots$

$$X \sim \text{Geo}(0.01)$$

Bullseye (3/4)

Example 3

We throw darts at a dartboard until we hit the bullseye. Assume throws are independent and the probability of hitting the bullseye is 0.01 for each throw.

2. What are the mean and variance for the number of throws needed to hit the bullseye?

$$E(X) = \frac{1}{p} = \frac{1}{0.01} = 100$$

$$\begin{aligned} \text{Var}(X) &= \frac{1-p}{p^2} = \frac{1-0.01}{0.01^2} = \frac{0.99}{0.0001} \\ &= 9,900 \end{aligned}$$

Bullseye (4/4)

Example 3

We throw darts at a dartboard until we hit the bullseye. Assume throws are independent and the probability of hitting the bullseye is 0.01 for each throw.

3. Find the probability that our first bullseye:

- (a) • is on one of the first fifty tries
- (b) • is after the 50th try, given that it did not happen on the first 20 tries

$$\begin{aligned}
 \textcircled{a} \quad P(X \leq 50) &= F_X(x=50) \\
 &= 1 - (1 - 0.01)^{50} \\
 &= 1 - 0.99^{50} = 0.395
 \end{aligned}$$

$$\textcircled{b} \quad P(X > 50 \mid X > 20) = \frac{P(X > 50, X > 20)}{P(X > 20)}$$

no hit in 1st 20

$$= \frac{P(X > 50)}{P(X > 20)}$$

$$= \frac{1 - P(X \leq 50)}{1 - P(X \leq 20)} = \frac{1 - F_X(50)}{1 - F_X(20)}$$

$$= \frac{1 - [1 - (1 - 0.01)^{50}]}{1 - [1 - (1 - 0.01)^{20}]} = \frac{0.99^{50}}{0.99^{20}} = 0.99^{30}$$

Memoryless property for Geometric RVs

If we know X is greater than some number (aka given $X > j$), then the probability of $X > k + j$ is just the probability that $X > k$.

\downarrow
20

$k + j = 50$
 $\downarrow \quad \downarrow$
30 20

$$P(X > k + j | X > j) = P(X > k)$$

$$P(X > k + j | X > j) = \frac{P(X > k + j \text{ and } X > j)}{P(X > j)} = \frac{P(X > k + j)}{P(X > j)} = \frac{(1 - p)^{k+j}}{(1 - p)^j} = (1 - p)^k$$

$$P(X > 50 | X > 20) = P(X > 30)$$

Negative Binomial RVs

Properties of Negative Binomial RVs

- **Scenario:** There are repeated independent trials, each resulting in a success or failure, with constant probability of success for each trial. We are counting the number of trials until the r^{th} success.
- Shorthand: $X \sim \text{NegBin}(p, r)$ or $X \sim \text{NB}(p, r)$
- Negative binomial is sum of r geometric distributions

$X =$ Number of independent trials until r^{th} success

$$p_X(x) = P(X = x) = \binom{x-1}{r-1} (1-p)^{x-r} p^r \text{ for } x = r, r+1, r+2, \dots$$

$x-r$ failures
 r succ

$$E(X) = \frac{r}{p}$$

$\rightarrow Y$ is geom(p)

$$E(X) = r \cdot E(Y)$$

$$\text{Var}(X) = \frac{rq}{p^2} = \frac{r(1-p)}{p^2}$$

$\rightarrow \text{Var}(x) = r \text{Var}(Y)$

Hitting more than 1 bullseye

Example 1

Consider again the bullseye example, where we throw darts at a dartboard until we hit the bullseye. Assume throws are independent and the probability of hitting the bullseye is 0.01 for each throw.

1. What is the expected value and variance of the number of throws needed to hit 5 bullseyes?

Hitting more than 1 bullseye

Example 1

Consider again the bullseye example, where we throw darts at a dartboard until we hit the bullseye. Assume throws are independent and the probability of hitting the bullseye is 0.01 for each throw.

2. What is the probability that the 5th bullseye is on the 20th throw?

$$p = 0.01 \quad r = 5$$

$X = \#$ trials to get 5 hits (bullseyes)

$$X \sim \text{NegBin}(r=5, p=0.01)$$

$$P(X=20) = \binom{20-1}{5-1} 0.99^{20-5} 0.01^5$$

ways to arrange the 1st 19 throws

$$= 0.0000033$$

$$3.3 \times 10^{-7}$$

5 minute break

Poisson RVs

Properties of Poisson RVs

- **Scenario:** We are counting the number of successes in a fixed time period (or fixed space), which has a constant rate (λ) of successes
- Shorthand: $X \sim \text{Poisson}(\lambda)$ or $X \sim \text{Pois}(\lambda)$

$X =$ Number of successes in a given period / given space

$$p_X(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} \text{ for } x = 0, 1, 2, 3, \dots$$

$$E(X) = \lambda$$

$$\text{Var}(X) = \lambda$$

Distinguishing between Binomial and Poisson RVs

- Recall that if $X \sim \text{Binomial}(n, p)$, then
 - X models the number of successes ...
 - in n independent (Bernoulli) trials ...
 - that each have the same probability of success p .
- Poisson RV's are similar,
 - except that instead of having n discrete independent trials,
 - there is a fixed time period (or space) during which the successes happen

★ probability

★ rate

Examples of Poisson RVs

- Number of visitors to an emergency room in an hour during a weekend night
- Number of study participants enrolled in a study per week
- Number of pedestrians walking through a square mile
- Any more?

Number of babies born w/ cleft pallet w/in 1 year

↳ rare events

Combining independent Poisson distributions

Theorem 1

If $X \sim \text{Pois}(\lambda_1)$ and $Y \sim \text{Pois}(\lambda_2)$ are independent of each other, then $Z = X + Y \sim \text{Pois}(\lambda_1 + \lambda_2)$.

Two emergency rooms

Example 2

Suppose emergency room 1 has an average of 50 visitors per day, and emergency room 2 has an average of 70 visitors per day, independently of each other. What is the probability distribution to model of the total number of visitors to both?

$$Y = \text{ER } 1$$

$$X = \text{ER } 2$$

$$Z = X + Y$$

$$Z \sim \text{Pois}(\lambda_1 + \lambda_2)$$

$$Z \sim \text{Pois}(50 + 70)$$

$$P(Z = z) = \frac{e^{-120} 120^z}{z!}$$

for $z = 0, 1, 2, \dots$

★ Poisson Approximation of the Binomial

Both Poisson and Binomial RV's are counting the number of successes

- If for a Binomial RV
 - the number of trials n is very large, and
 - the probability of success p is close to 0 or 1,
- Then the Poisson distribution can be used to approximate Binomial probabilities
 - and we use $\lambda = np$
- **Rule of thumb:** We can use the Poisson approximation when $\frac{1}{10} \leq np(1-p) \leq 10$

Medical lab errors

Example 3

Suppose that in the long run, errors in a medical testing lab are made 0.1% of the time. Find the probability that fewer than 4 mistakes are made in the next 2,000 tests.

1. Find the probability using the Binomial distribution.
2. Approximate the probability in part (1) using the Poisson distribution.

To do for extra practice - will also see a similar problem in BSTA 511

Hypergeometric RVs

Hypergeometric RVs

- **Scenario:** There are a fixed number of successes and failures (which are known in advance), from which we make n draws without replacement. We are counting the number of successes from the n trials.
 - There is a finite population of N items
 - Each item in the population is either a success or a failure, and there are M successes total.
 - We randomly select (sample) n items from the population without replacement
- Shorthand: $X \sim \text{Hypergeo}(M, N, n)$

n draws from N items
w/in N we have M successes & $N-M$ fail

$X =$ Number of successes in n draws

$$E(X) = \frac{nM}{N}$$

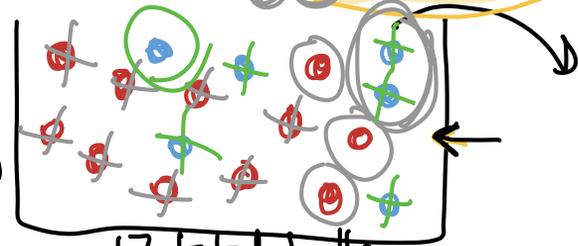
$$p_X(x) = P(X=x) = \frac{\binom{M}{x} \binom{N-M}{n-x}}{\binom{N}{n}}$$

succ (pointing to $\binom{M}{x}$) *failures* (pointing to $\binom{N-M}{n-x}$) *all draws* (pointing to $\binom{N}{n}$)

$$\text{Var}(X) = n \frac{M}{N} \left(1 - \frac{M}{N}\right) \left(\frac{N-n}{N-1}\right)$$

for x integer-valued $\max(0, n - (N - M)) \leq x \leq \min(n, M)$

blue = success
 $N = 17$
 $M = 6$
 $N - M = 11$



$n = 5$ draws

$P(X=x)$
blue balls draw out of 5

Wolf population

Example 4

A wildlife biologist is using mark-recapture to research a wolf population. Suppose a specific study region is known to have 24 wolves, of which 11 have already been tagged. If 5 wolves are randomly captured, what is the probability that 3 of them have already been tagged?

↳ success

let $X = \#$ of wolves tagged out of 5

$$N = 24$$

$$M = 11$$

$$N - M = 13$$

$$n = 5$$

} parameters

looking for $P(X=3)$

$$P(X=x) = \frac{\binom{M}{x} \binom{N-M}{n-x}}{\binom{N}{n}} = \frac{\binom{11}{x} \binom{13}{5-x}}{\binom{24}{5}}$$

$$P(X=3) = \frac{\binom{11}{3} \binom{13}{2}}{\binom{24}{5}}$$

R command:
choose (top, bottom)

Binomial approximation of the hypergeometric RV

Suppose a hypergeometric RV X has the following properties:

- the population size N is really big,
- the number of successes M in the population is relatively large,
▪ $\frac{M}{N}$ shouldn't be close to 0 or 1
- and the number of items n selected is small (relative to N)
- Rule of thumb: $\frac{n}{N} < 0.05$ or $N > 20n$

$\frac{M}{N}$ is like p
in Binomial
Bin has constant p

Then, in this case, making n draws from the population doesn't change the probability of success much, and the hypergeometric RV. can be approximated by a binomial RV \rightarrow w/ approx $p = \frac{M}{N}$

$n = 5$

$$\begin{array}{r} 1100 \\ \hline 2400 \\ \hline 0.458 \end{array} \quad \rightarrow \quad \begin{array}{r} 1095 \\ \hline 2395 \\ \hline 0.457 \end{array}$$

Wolf population revisited

Example 5

Suppose a specific study region is known to have 2400 wolves, of which 1100 have already been tagged.

1. If 50 wolves are randomly captured, what is the probability that 20 of them have already been tagged?
2. Approximate the probability in part (1) using the binomial distribution.

① $n = 50$ $X = \#$ of tagged wolves in 50
 $X \sim \text{Hypergeom}(N=2400, M=1100, n=50)$

$$P(X=20) = \frac{\binom{1100}{20} \binom{1300}{30}}{\binom{2400}{50}} = 0.08132$$

hyper geom

② check $N > 20n$ $2400 > 20(50) = 1000$ ✓

$$\frac{M}{N} = \frac{1100}{2400} = \frac{11}{24} = 0.458 \checkmark$$

$\frac{M}{N}$ vs. $\frac{M-n}{N-n}$ starting p vs ending p .

binom

$$X = \# \text{ of tagged in } 50$$

$$X \sim \text{Binom}(n=50, p = \frac{M}{N} = \frac{11}{24})$$

$$P(X=20) = \binom{50}{20} \left(\frac{11}{24}\right)^{20} \left(\frac{13}{24}\right)^{30} = 0.08107$$

Discrete Uniform RVs

Discrete Uniform RVs

- **Scenario:** There are N possible outcomes, which are all equally likely.
- Shorthand: $X \sim \text{Uniform}(N)$

$X =$ Outcome of interest, with $x = \overset{\downarrow}{\overset{\downarrow}{\underline{1, 2, \dots, N}}}$

$$p_X(x) = P(X = x) = \frac{1}{N} \text{ for } x = 1, 2, 3, \dots, N$$

$$E(X) = \frac{N + 1}{2}$$

$$\text{Var}(X) = \frac{N^2 - 1}{12}$$

What discrete uniform RVs have we seen already?

Example 6

Examples of discrete uniform RVs

① Rolling fair die $N = 6$

② Toss coin $N = 2$

③ Pulled cards from a deck
 $N = 52$

④ Pulled ♡ from deck $N = 4$
suits

sample ()

All the R code for these!

Check out [this page](#) with all the different functions for distributions in R

Example of R commands for hypergeometric distribution with their **input** and **output**:

R code	What does it return?
<code>rhyper()</code>	returns sample of random variables with specified distribution
<code>dhyper()</code>	returns value of probability density at certain point of the distribution
<code>phyper()</code>	returns cumulative probability of getting certain point (or less) of the normal distribution
<code>qhyper()</code>	returns inverse CDF corresponding to desired quantile