Chapter 1: Outcomes, Events, and Sample Spaces

Meike Niederhausen and Nicky Wakim

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Learning Objectives

1. Define basic terms related to events such as events, outcomes, and sample space.
2. Use proper set notation for events
3. Characterize possible outcomes, when something random occurs
4. Describe events into which outcomes can be grouped
5. Define important terms and rules within set theory such as unions, intersections, complements, mutually exclusive, and De Morgan's Laws

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Probability axioms
- Probability properties
- Counting
- Independence
- Conditional probability
- Bayes' Theorem
- Random Variables

Probability for discrete random variables

- Functions: pmfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Probability for continuous random variables

- Calculus
- Functions: pdfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

Tossing One Coin (Outcomes, Events, and Sample Space)

Coin Toss Example: 1 coin (1/3)

Suppose you toss one coin.

- What are the possible outcomes?
- What is the sample space?
- What are the possible events?

Coin Toss Example: 1 coin (2/3)

Suppose you toss one coin.

- What are the possible outcomes?
  - Heads (H)
  - Tails (T)

Note

When something happens at random, such as a coin toss, there are several possible outcomes, and *exactly one* of the outcomes will occur.

not necessarily event

Definitions: Sample Space and Events

Definition: Sample Space

The **sample space** S is the set of *all* outcomes

Definition: Event

An **event** is a *collection of some* outcomes. An event can include multiple outcomes or no outcomes (a subset of the sample space).

When thinking about events, think about outcomes that you might be asking the probability of. For example, what is the probability that you get a heads and a tails in one flip? (Answer: 0)

Coin Toss Example: 1 coin (3/3)

- What is the sample space?
  - $S = \{H, T\}$
- What are the possible events?
  - H
  - T
  - {H, T}
  - ∅ (empty set)

$2^2 = 4$

Note #1

We use curly brackets ($\{\}$) to denote a set (collecting a list of outcomes or values)

Note #2

The total number of possible events is

$$2^{|S|}$$

where $|S|$ is the total number of outcomes in the sample space. Also, possible events are not necessarily something that can actually occur (i.e. getting a heads and a tails on a single coin flip)

Tossing Two Coins (Outcomes, Events, and Sample Space)

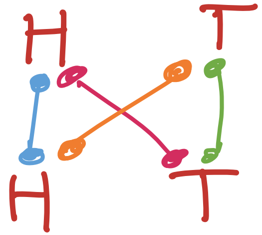
Coin Toss Example: 2 coins

Suppose you toss two coins.

- What is the sample space? *Assume the coins are distinguishable*
  - $S = \{HH, TT, HT, TH\}$
- What are some possible events?
  - A = exactly one H = $\{HT, TH\}$
  - B = at least one H = $\{HH, HT, TH\}$
  -
  - C = no tails = {HH}
- D = ϕ → $P(\phi) = 0$

coin 1

coin 2



$2^{|S|}$ → # outcomes in S

$2^4 = 16$

{HH} {HT} {TH} {TT}

{HT, TH} {HH, TT} ...

{HT, TH, HH} ....

{HT, TH, HH, TT}

included in event	HH	TT	HT	TH
	Y	Y	Y	Y
	N	N	N	N

→ 2

More info on events and sample spaces

- We usually use capital letters from the beginning of the alphabet to denote events. However, other letters might be chosen to be more descriptive.

- We use the notation $|S|$ to denote the **size** of the sample space.

- The total number of possible events is $2^{|S|}$, which is the total number of possible subsets of S.

- The **empty set**, denoted by $\emptyset$, is the set containing no outcomes.

Example: Keep sampling until...

Suppose you keep sampling people until you have someone with high blood pressure (BP)

What is the sample space?

- Let H = denote someone with high BP.
- Let H^C = denote someone with not high blood pressure, such as low or regular BP.

  H'
- Then, $S = \{H, H^cH, H^cH^cH, H^cH^cH^cH, \ldots\}$

$|S| = \infty$

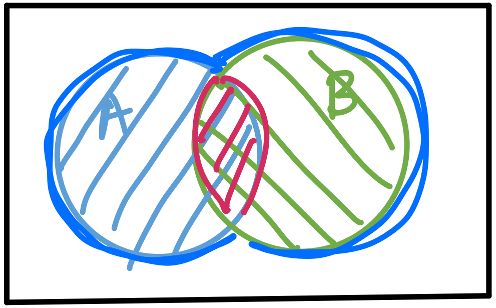
Set Theory

Set Theory (1/2)

Venn diagrams

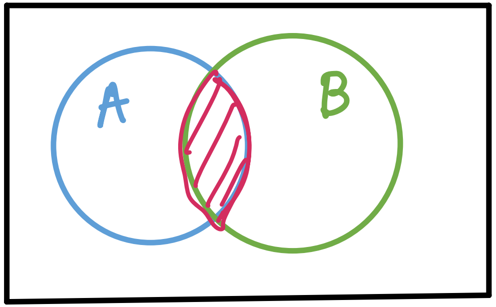
Definition: Union

The **union** of events A and B, denoted by $A \cup B$, contains all outcomes that are in A *or* B or both



Definition: Intersection

The **intersection** of events A and B, denoted by $A \cap B$, contains all outcomes that are both in A *and* B.



Set Theory (2/2)

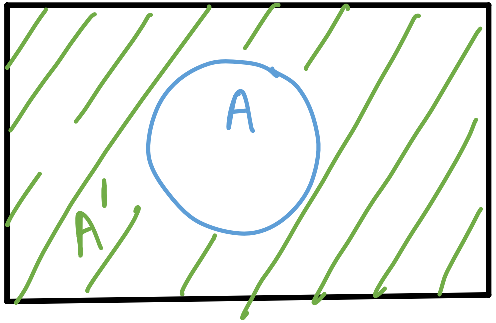
Venn diagrams

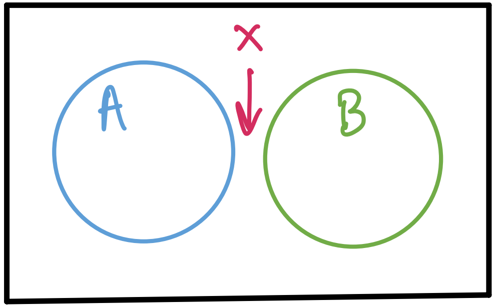
Definition: Complement

The **complement** of event A, denoted by A^C or A', contains all outcomes in the sample space S that are *not* in A .

Definition: Mutually Exclusive

Events A and B are **mutually exclusive**, or disjoint, if they have no outcomes in common. In this case $A \cap B = \emptyset$, where $\emptyset$ is the empty set.





BP example variation (1/3)

- Suppose you have n subjects in a study.
- Let H_i be the event that person i has high BP, for $i = 1 \ldots n$.

Use set theory notation to denote the following events:

1. Event subject i does not have high BP
2. Event all n subjects have high BP
3. Event at least one subject has high BP
4. Event all of them do not have high BP
5. Event at least one subject does not have high BP

BP example variation (2/3)

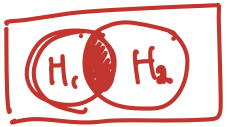
- Suppose you have n subjects in a study.
- Let H_i be the event that person i has high BP, for $i = 1 \dots n$.

Use set theory notation to denote the following events:

1. Event subject i does not have high BP H_i' or H_i^c

2. Event all n subjects have high BP $H_1 \cap H_2$

P1 & P2: H_1 H_2

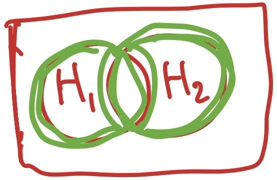


$H_1 \cap H_2 \cap H_3 \cap \dots \cap H_n = \bigcap_{i=1}^{n} H_i$

$\sum_{i=1}^{n} X_i$ $X_1 + X_2 + \dots X_n$

3. Event at least one subject has high BP

P1 & P2



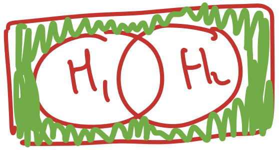
$H_1 \cup H_2$

$H_1 \cup H_2 \cup H_3 \cup \dots \cup H_n = \bigcup_{i=1}^{n} H_i$

BP example variation (3/3)

4. Event all of them do not have high BP

P1 & P2



$H_1^c \cap H_2^c$

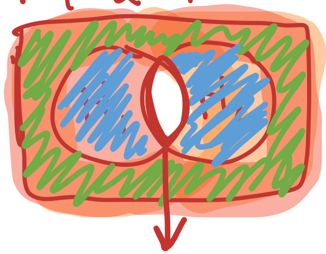
$(H_1 \cup H_2)^c$

$H_1^c \cap H_2^c \cap H_3^c \cap \dots \cap H_n^c = \bigcap_{i=1}^{n} H_i^c$

$(H_1 \cup H_2 \cup H_3 \cup \dots \cup H_n)^c = \left(\bigcup_{i=1}^{n} H_i\right)^c$

5. Event at least one subject does not have high BP

P1 & P2



$(H_1 \cap H_2)^c$

$H_1^c \cup H_2^c$

$(H_1 \cap H_2 \cap H_3 \cap \dots \cap H_n)^c = \left(\bigcap_{i=1}^{n} H_i\right)^c$

$H_1^c \cup H_2^c \cup H_3^c \cup \dots \cup H_n^c = \bigcup_{i=1}^{n} H_i^c$

De Morgan's Laws

Theorem: De Morgan's 1st Law

For a collection of events (sets) $A_1, A_2, A_3, \ldots$

$$\bigcap_{i=1}^{n} A_i^C = \left(\bigcup_{i=1}^{n} A_i\right)^C$$

"all not A = (at least one event A)^C" or "intersection of the complements is the complement of the union"

Theorem: De Morgan's 2nd Law

For a collection of events (sets) $A_1, A_2, A_3, \ldots$

$$\bigcup_{i=1}^{n} A_i^C = \left(\bigcap_{i=1}^{n} A_i\right)^C$$

"at least one event not A = (all A)^C" or "union of complements is complement of the intersection"

Remarks on De Morgan's Laws

- These laws also hold for infinite collections of events. $n = \infty$

- Draw Venn diagrams to convince yourself that these are true!

- These laws are *very* useful when calculating probabilities.
  - This is because calculating the probability of the **intersection of events is often much easier than the union of events**.
  - This is not obvious right now, but we will see in the coming chapters why.