Chapter 2: Introduction to Probability

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Learning Objectives

1. Define basic axioms and propositions in probability
2. Assign probabilities to events
3. Perform manipulations on probabilities to make calculations easier

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Probability axioms
- Probability properties
- Counting
- Independence
- Conditional probability
- Bayes' Theorem
- Random Variables

Probability for discrete random variables

- Functions: pmfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Probability for continuous random variables

- Calculus
- Functions: pdfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

Probabilities of equally likely events

Probabilities of equally likely events

- "Equally likely" means the probability of any possible outcome is the same
  - Think: each side of die is equally likely or picking a card in a deck is equally likely

Pick an *equally likely* card, any *equally likely* card

Example 1

Suppose you have a regular well-shuffled deck of cards. What's the probability of drawing:

1. any heart
2. the queen of hearts
3. any queen

Sample space: 52 cards

$\underline{4}$ suits

$\underline{13}$ #s/faces: w/in each suit

① $P(\heartsuit) = \frac{13}{52} = \frac{1}{4} = 0.25$

② $P(Q\heartsuit) = \frac{1}{52}$

$P(\ ) \rightarrow$ "probability of"

③ $P(Q) = \frac{4}{52} = \frac{1}{13}$

Let's break down this probability

If S is a finite sample space, with **equally likely outcomes**, then

$$\mathbb{P}(A) = \frac{|A|}{|S|}$$

In human speak:

- For equally likely outcomes, the probability that a certain event occurs is: the number of outcomes within the event of interest ($|A|$) **divided by** the total number of possible outcomes ($|S|$)

$$\mathbb{P}(A) = \frac{\text{total number of outcomes in event A}}{\text{total number of outcomes in sample space}}$$

- Thus, it is important to be able to count the outcomes within an event

A probability is a function...

- $\mathbb{P}(A)$ is a function with
  - **Input:** event A from the sample space S, ($A \subseteq S$)
    - $A \subseteq S$ means "A contained within S" or "A is a subset of S"
  - **Output:** a number between 0 and 1 (inclusive)

- The **probability function** maps an event (input) to value between 0 and 1 (output)
  - When we speak of the probability function, we often call the values between 0 and 1 "probabilities"
    - Example: "The probability of drawing a heart is 0.25" for $P(\text{heart}) = 0.25$

- The probability function needs to follow some specific rules!

See Probability Axioms on next slide.

Probability Axioms

Probability Axioms

Axiom 1

For every event A, $0 \leq \mathbb{P}(A) \leq 1$. Probability is between 0 and 1.

Axiom 2

For the sample space S, $\mathbb{P}(S) = 1$.

$$P(S) = \frac{|S|}{|S|} = 1$$

Axiom 3

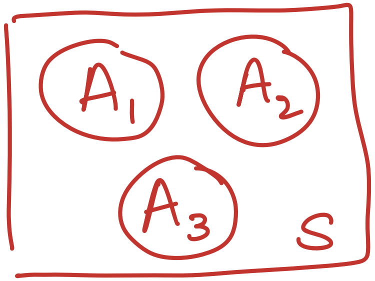
If $A_1, A_2, A_3, \ldots$, is a collection of disjoint events, then

$$A_1 \cap A_2 = \emptyset$$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The probability of at least one A_i is the sum of the individual probabilities of each.

$A_1 \quad A_2 \quad A_3$



$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$$

Some probability properties

Some probability properties

Using the Axioms, we can prove all other probability properties! Events A, B, and C are not necessarily disjoint!

Proposition 1

For any event A, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

Proposition 2

$\mathbb{P}(\emptyset) = 0$

Proposition 3

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

Proposition 4

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

where A and B are not necessarily disjoint

Proposition 5

$$\begin{aligned}\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \\ \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \\ \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)\end{aligned}$$

Proposition 1 Proof

Proposition 1

For any event A, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

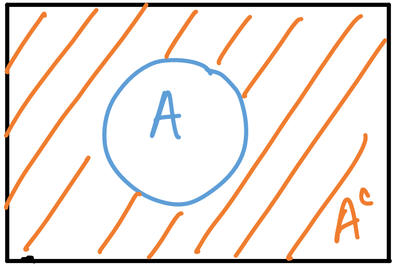
Use Axioms!

A1: $0 \leq \mathbb{P}(A) \leq 1$

A2: $\mathbb{P}(S) = 1$

A3: For disjoint A_i,

$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$



$A \cup A^c = S$

A & A^c are disjoint

$P(A \cup A^c) = P(S) = \underline{1}$

$P(A \cup A^c) = P(A) + P(A^c)$

$= 1$

$1 = P(A) + P(A^c)$

$-P(A^c) \qquad - P(A^c)$

$P(A) = 1 - P(A^c)$

Proposition 2 Proof

Proposition 2

$\mathbb{P}(\emptyset) = 0$

Use Axioms!

A1: $0 \leq \mathbb{P}(A) \leq 1$

A2: $\mathbb{P}(S) = 1$

A3: For disjoint A_i,

$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

prop 1 : $\underline{P(A)} = 1 - \underline{P(A^c)}$

let $A = \phi$ $\quad A^c = S$

$$\underline{P(\phi)} = 1 - \underbrace{\underline{P(S)}}_{= 1}$$

$$= 1 - 1$$

$P(\phi) -$

Proposition 3 Proof

Proposition 3

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

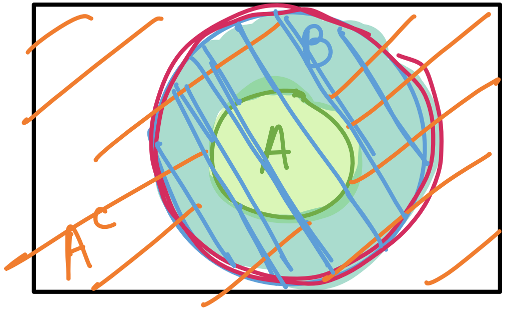
Use Axioms!

A1: $0 \leq \mathbb{P}(A) \leq 1$

A2: $\mathbb{P}(S) = 1$

A3: For disjoint A_i,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

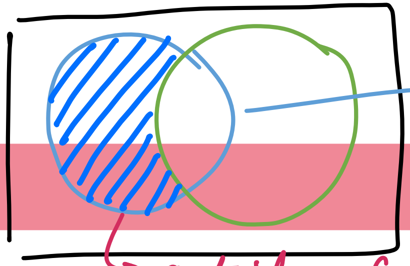


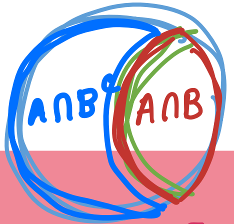
$$B = A \cup (B \cap A^c)$$

$$\underline{P(B)} = \underline{P(A)} + \underbrace{P(B \cap A^c)}_{\geq 0}$$

$$P(B) \geq P(A)$$

Proposition 4 Visual Proof



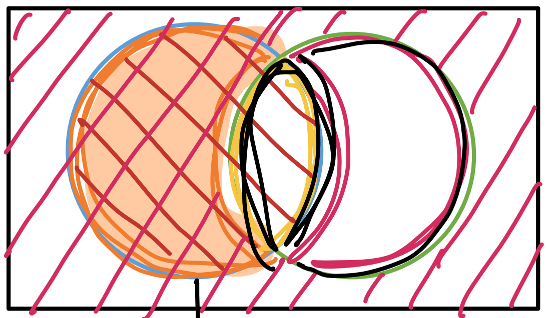


Proposition 4

$$\mathbb{P}(A\cup B)=\mathbb{P}(A)+\mathbb{P}(B)-\mathbb{P}(A\cap B)$$

outside of B (B^c) in A still

$$A = [A\cap B^c] \cup [A\cap B]$$

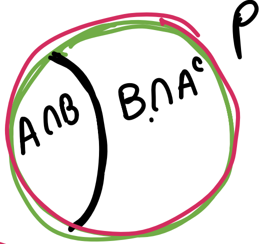


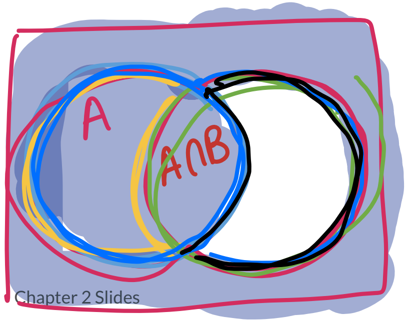
$$A\cup B = [A\cap B^c]\cup[A\cap B]\cup[B\cap A^c]$$

$$P(A\cup B) = \underbrace{P(A\cap B^c)}_{P(A)-\cancel{P(A\cap B)}} + \cancel{P(A\cap B)} + \underbrace{P(B\cap A^c)}_{P(B)-P(B\cap A)}$$

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)$$

$A\cap B^c$



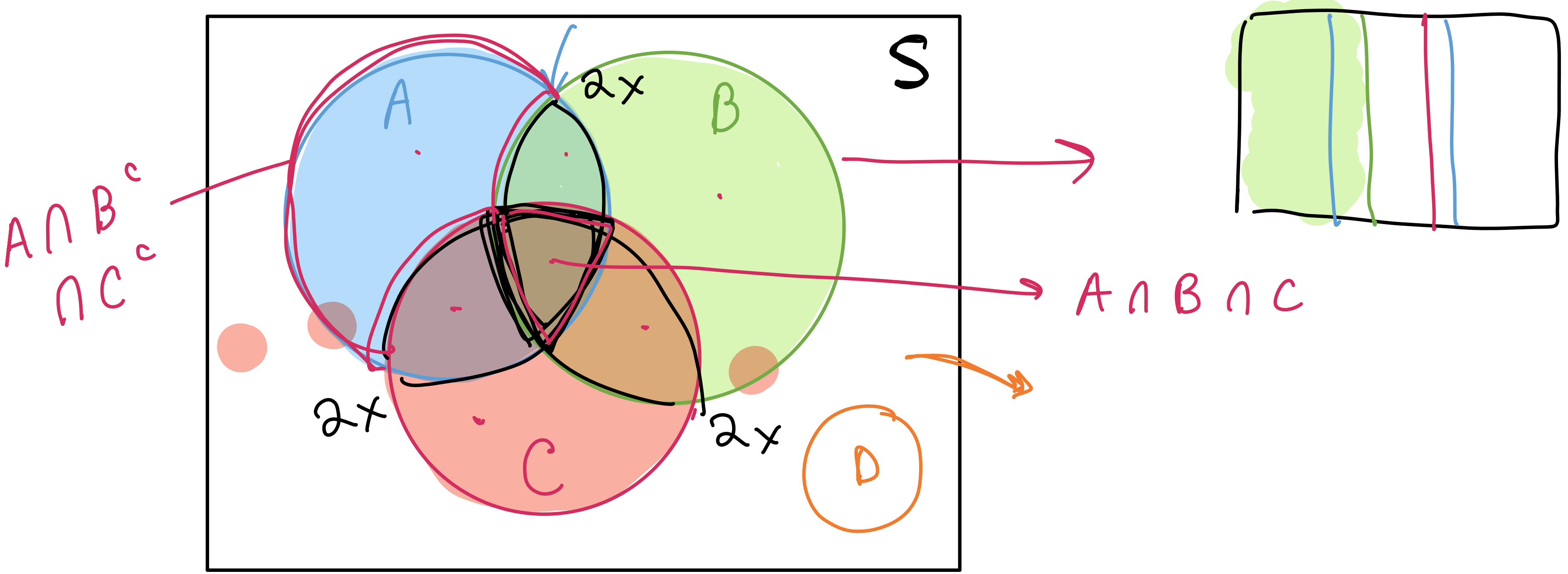


$A\cup B^c =$ $B\cap A^c$

Proposition 5 Visual Proof

Proposition 5

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$$



Some final remarks on these proposition

- Notice how we spliced events into multiple **disjoint** events
  - It is often easier to work with disjoint events
- If we want to calculate the probability for one event, we may need to get creative with how we manipulate other events and the sample space
  - Helps us use any incomplete information we have

Partitions

Partitions

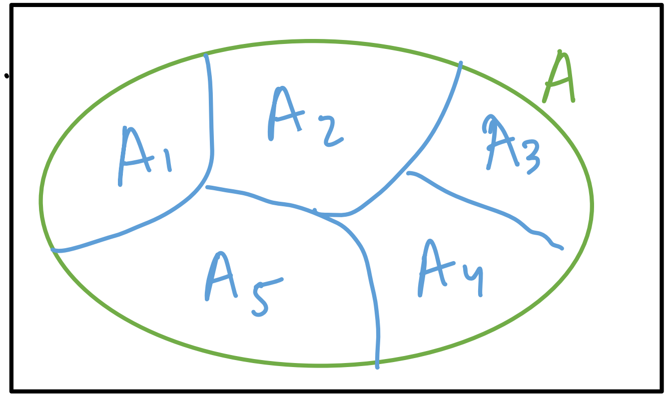
$\{A_1, A_2, A_3, \ldots A_n\}$

$A = (A \cap B^c) \cup (A \cap B)$

Definition: Partition

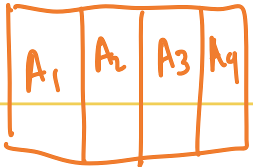
A set of events $\{A_i\}_{i=1}^{n}$ create a **partition** of A, if

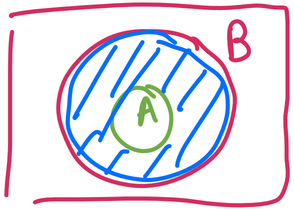
- the A_i's are disjoint (mutually exclusive) and
- $\bigcup_{i=1}^{n} A_i = A$



Example 2

- If $A \subset B$, then $\{A, B \cap A^C\}$ is a partition of B. → prop 3
- If $S = \bigcup_{i=1}^{n} A_i$, and the A_i's are disjoint, then the A_i's are a partition of the sample space. →





$\{A, B \cap A^c\}$

Creating partitions is sometimes used to help calculate probabilities, since by Axiom 3 we can add the probabilities of disjoint events.

Venn Diagram Probabilities

Weekly medications

Example 3

If a subject has an

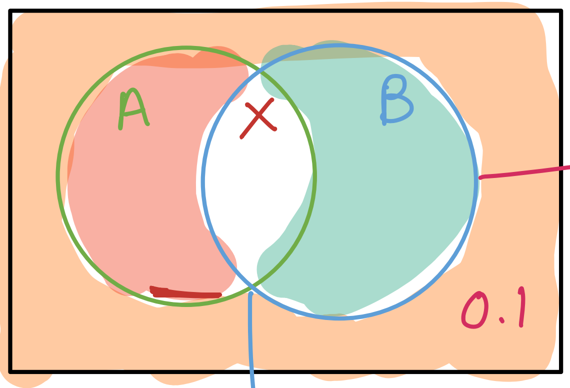
- 80% chance of taking their medication *this* week, $P(A) = 0.8$
- 70% chance of taking their medication *next* week, and $P(B) = 0.7$
- 10% chance of *not* taking their medication *either* week, $P((A \cup B)^c) = 0.1$

then find the probability of them taking their medication exactly one of the two weeks.

Hint: Draw a Venn diagram labelling each of the parts to find the probability.

Let A = take med this week

B = take med NEXT week



$P(A \cup B) = 1 - P((A \cup B)^c) = 0.9$

$P(A^c \cap B^c) = 0.1$

$P(A \cap B^c) + P(B \cap A^c)$

$\underbrace{P(A \cup B)}_{0.9} - P(A \cap B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$0.9 = \underline{0.8 + 0.7} - P(A \cap B)$$

$$P(A \cap B) = 0.6$$

$$P(A \cup B) - P(A \cap B)$$

$$= 0.9 - 0.6 = \underline{0.3}$$

The prob of them taking their med exactly one of the weeks is 0.3.