Chapter 3: Independent Events

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Learning objectives

1. Define independence of 2-3 events given probability notation
2. Calculate whether two or more events are independent

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Probability axioms
- Probability properties
- Counting
- Independence
- Conditional probability
- Bayes' Theorem
- Random Variables

Probability for discrete random variables

- Functions: pmfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Probability for continuous random variables

- Calculus
- Functions: pdfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

Independent Events

Definition: Independence

Events A and B are **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Notation: For shorthand, we sometimes write $A \perp\!\!\!\perp B$, to denote that A and B are independent events.

- Also note:

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B) \implies A \perp\!\!\!\perp B$$
$$A \perp\!\!\!\perp B \implies \mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

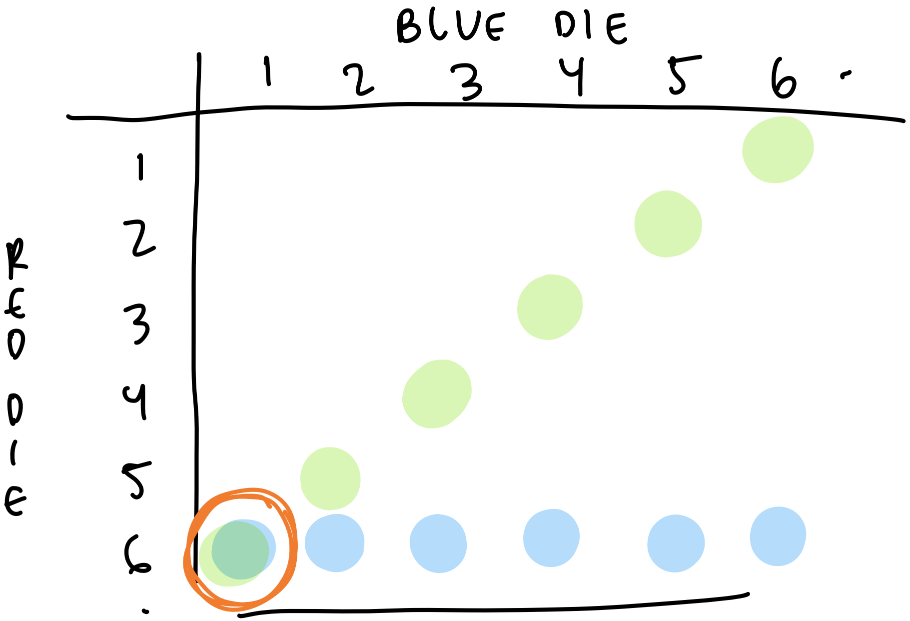
Example of two dice

Example 1

Two dice (red and blue) are rolled. Let A = event a total of 7 appears, and B = event red die is a six. Are events A and B independent?

$$P(A \cap B) \stackrel{?}{=} P(A)P(B)$$

$$|S| = 6 \times 6 = 36$$



$$P(A) = \frac{|A|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

$$P(B) = \frac{|B|}{|S|} = \frac{6}{36} = \frac{1}{6}$$

$$P(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{1}{36}$$

$$=$$

$$P(A)P(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

$$P(A \cap B) = \frac{1}{36} = P(A)P(B) \Rightarrow A \perp B$$

Independence of 3 Events

Definition: Independence of 3 Events

Events A, B, and C are *mutually* **independent** if $\longrightarrow$ $A \perp B$, $B \perp C$, $A \perp C$

$A \perp B \perp C$

1. • $\mathbb{P}(A \cap B) = \underline{\mathbb{P}(A) \cdot \mathbb{P}(B)}$
   • $\mathbb{P}(A \cap C) = \underline{\mathbb{P}(A) \cdot \mathbb{P}(C)}$
   • $\mathbb{P}(B \cap C) = \underline{\mathbb{P}(B) \cdot \mathbb{P}(C)}$
2. $\underline{\mathbb{P}(A \cap B \cap C)} = \underline{\mathbb{P}(A)} \cdot \underline{\mathbb{P}(B)} \cdot \underline{\mathbb{P}(C)}$

Remark:

On your homework you will show that $(1) \not\Rightarrow (2)$ and $(2) \not\Rightarrow (1)$.

Probability at least one smoker

Example 2

Suppose you take a random sample of n people, of which people are smokers and non-smokers independently of each other. Let

- A_i = event person i is a smoker, for $i = 1, \ldots, n$, and
- p_i = probability person i is a smoker, for $i = 1, \ldots, n$.

Find the probability that at least one person in the random sample is a smoker.

$$\bigcup_{i=1}^{n} A_i^c = \left(\bigcap A_i\right)^c$$

$$\bigcup A_i = \left(\bigcap A_i^c\right)^c$$

$$A_i \perp A_j \quad i \neq j \quad i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, n$$

$$p_i = P(A_i) \qquad \underline{P(A_i) + P(A_i^c)} = 1 \rightarrow P(B) + P(B^c) = 1$$

$$= 1 - P(B^c)$$

$$P(\text{at least one smoker}) = P\Big(\underbrace{\bigcup_{i=1}^{n} A_i}_{B}\Big)$$

$$= P\Big(\underbrace{\Big(\bigcap_{i=1}^{n} A_i^c\Big)^c}_{B}\Big)$$

$$= 1 - P\Big(\underbrace{\bigcap_{i=1}^{n} A_i^c}_{B^c}\Big)$$

$$= 1 - P(A_1^c)P(A_2^c) \cdots P(A_n^c)$$

$$A_1 \cup A_2 \cup A_3 \ldots \cup A_n$$

De Morgan's law $\boxed{\bigcup_{i=1}^{n} A_i} = \left(\bigcap_{i=1}^{n} A_i^c\right)^c \rightarrow P(A \cap B) = P(A)P(B)$

$$= 1 - \prod_{i=1}^{n} P(A_i^c)$$

extension:

if $\underline{p_i = p}$

$$1 - \prod_{i=1}^{n} (1-p) = \underline{1 - (1-p)^n}$$

$n \to \infty$

$0 \leq p \leq 1$

$0 \leq 1-p \leq 1$

$$(1-p) \xrightarrow[n\to\infty]{} 0$$

$$P(at \mid smk) \xrightarrow[n\to\infty]{} 1$$