Chapter 4: Conditional Probability

Meike Niederhausen and Nicky Wakim

2024-10-09

Learning Objectives

1. Use set process to calculate probability of event of interest
2. Calculate the probability of an event occurring, given that another event occurred.
3. Define keys facts for conditional probabilities using notation.

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Probability axioms
- Probability properties
- Counting
- Independence
- Conditional probability
- Bayes' Theorem
- Random Variables

Probability for discrete random variables

- Functions: pmfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Probability for continuous random variables

- Calculus
- Functions: pdfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

General Process for Probability Word Problems

1. Clearly define your events of interest
2. Translate question to probability using defined events OR Venn Diagram
3. Ask yourself:
   - Are we sampling with or without replacement?
   - Does order matter?
4. Use axioms, properties, partitions, facts, etc. to define the end probability calculation into smaller parts
   - If probabilities are given to you, Venn Diagrams may help you parse out the events and probability calculations
   - If you need to find probabilities with counting, pictures or diagrams might help here
5. Write out a concluding statement that gives the probability context
6. (For own check) Make sure the calculated probability follows the axioms. Is is between 0 and 1?

Let's revisit our deck of cards

Example 1

Suppose we randomly draw 2 cards from a standard deck of cards. What is the probability that we draw a spade then a heart?

① Let

- Let A = event 1^{st} card is spade
- Let B = event 2^{nd} card is heart

② $P(A \cap B) = ?$

③ sampling rep? N

order matter? Y

④ counting: $P(A \cap B) = \frac{|A \cap B|}{|S|}$ $\quad \frac{13}{52} \cdot \frac{13}{51}$

$= \frac{13 \cdot 13}{52 \cdot 51} = 0.064$

$P(A \cap B) = \underbrace{P(A)}_{\text{1st spade}} \cdot \underbrace{P(B|A)}_{\text{2nd ♡ given 1st spade}}$

$= \left(\frac{13}{52}\right) \cdot \left(\frac{13}{51}\right)$

cannot use $P(A \cap B) = P(A)P(B)$

b/c $A \not\perp B$

Conditional Probability facts (1/2)

Fact 1: General Multiplication Rule

$$\underline{\mathbb{P}(A \cap B)} = \underline{\mathbb{P}(A)} \cdot \underline{\mathbb{P}(B|A)}$$

also $P(A \cap B) = P(B)\,P(A|B)$

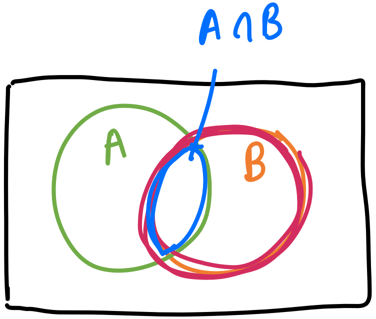
Fact 2: Conditional Probability Definition

$$\underline{\mathbb{P}(A|B)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

if $A \perp B$ $\quad P(A|B) = P(A)$

$$= \frac{P(A \cap B)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}}$$



Conditional Probability facts (2/2)

$P(B|A) + P(B^c|A) = 1$

Fact 3

If A and B are independent events ($A \perp\!\!\!\perp B$), then

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$

Fact 4

$\mathbb{P}(A|B)$ is a probability, meaning that it satisfies the probability axioms. In particular,

$$\mathbb{P}(A|B) + \mathbb{P}(A^C|B) = 1$$

if $A \perp B$ $P(A|B) = P(A)$

$$\hookrightarrow = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\cancel{P(B)}}{\cancel{P(B)}}$$

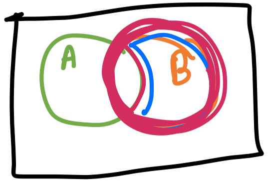
$$P(A|B) + P(A^c|B) = 1$$

$$\frac{P(A \cap B)}{P(B)} + \frac{P(A^c \cap B)}{P(B)} = 1$$

$$\rightarrow P(B)$$

$$P(A \cap B) + P(A^c \cap B) = P(B)$$

$\hookrightarrow$

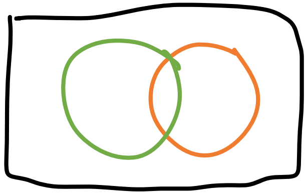


* find in prev notes

NOT RELATED

$P(A|B) + P(A|B^c)$

$$\frac{P(A \cap B)}{P(B)} + \frac{P(A \cap B^c)}{P(B)^c}$$



Monty Hall Problem

Survivor Season 42

With the Wiki page on it!

Conditional probability with two dice ④

Example 2

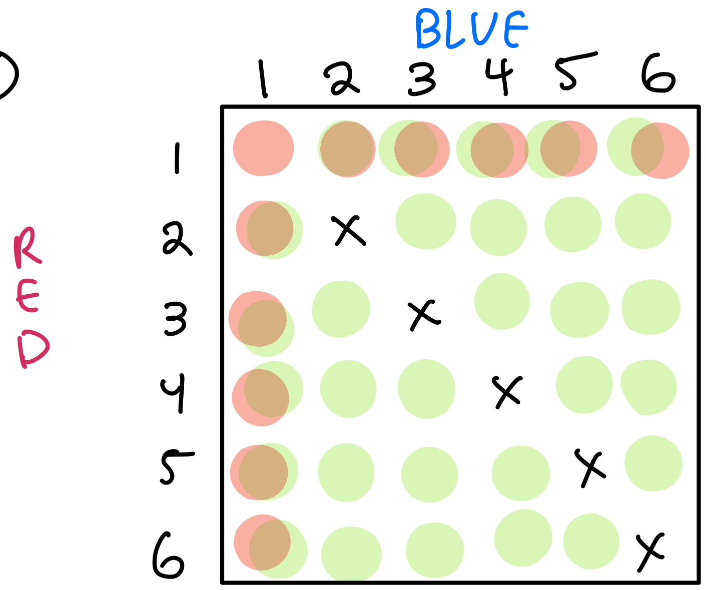
Two dice (red and blue) are rolled. If the dice do not show the same face, what is the probability that one of the dice is a 1?

① Let A = one die is a 1

B = do not show same face

② $P(A \mid B)$

③ N/A



6 by 6 grid = 36 options

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{10}{\cancel{36}}}{\frac{30}{\cancel{36}}} = \frac{10}{30} = \frac{1}{3}$$

⑤ The probability that one die is a 1 given they do not show the same face is $\frac{1}{3}$.