Chapter 5: Bayes' Theorem

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Learning Objectives

1. Calculate conditional probability of an event using Bayes' Theorem
2. Utilize additional probability rules in probability calculations, specifically the Higher Order Multiplication Rule and the Law of Total Probabilities

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Probability axioms
- Probability properties
- Counting
- Independence
- Conditional probability
- Bayes' Theorem
- Random Variables

Probability for discrete random variables

- Functions: pmfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Probability for continuous random variables

- Calculus
- Functions: pdfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

Introduction

- So we learned about conditional probabilities
  - We learned how the occurrence of event A affects event B (B conditional on A)
- Can we figure out information on how the occurrence of event B affects event A?
- We can use the conditional probability ($\mathbb{P}(\mathrm{A}|\mathrm{B})$) to get information on the flipped conditional probability ($\mathbb{P}(\mathrm{B}|\mathrm{A})$)

$A|B$

$B|A$

Bayes' Rule for two events

Theorem: Bayes' Rule (for two events)

For any two events A and B with nonzero probabilties,

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A) \cdot \mathbb{P}(B|A)}{\mathbb{P}(B)}$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A)P(B|A)$$

Calculating probability with Higher Order Multiplication Rule

Example 1

Suppose we draw 5 cards from a standard shuffled deck of 52 cards. What is the probability of a flush, that is all the cards are of the same suit (including straight flushes)?

Higher Order Multiplication Rule

$$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2 | A_1) \cdot \mathbb{P}(A_3 | A_1 A_2) \ldots \cdot \mathbb{P}(A_n | A_1 A_2 \ldots A_{n-1})$$

$P(A \cap B \cap C) = P(A)\, P(B|A) \cdot P(C | B, A)$

$P(A \cap B)$

① A_1 = card of any suit

A_i = get same card as A_1

$i = 2, 3, 4, 5$

② $P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)$

③ order matters, no replacement

$P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = \left(\frac{52}{52}\right)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right)\left(\frac{10}{49}\right)\left(\frac{9}{48}\right)$

$= 0.00198$

④ $P(A_1) = \frac{52}{52}$ $\quad P(A_2 | A_1) = \frac{12}{51}$ $\quad P(A_5 | A_1 - A_4) = \frac{9}{48}$

$P(A_3 | A_1, A_2) = \frac{11}{50}$ $\quad P(A_4 | A_1, A_2, A_3) = \frac{10}{49}$

⑤ The prob of a flush is 0.00198.

Calculating probability with Law of Total Probability

Example 2

Suppose 1% of people assigned female at birth (AFAB) and 5% of people assigned male at birth (AMAB) are color-blind. Assume person born is equally likely AFAB or AMAB (not including intersex). What is the probability that a person chosen at random is color-blind?

Law of Total Probability for 2 Events

For events A and B,

$$\mathbb{P}(B) = \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^C)$$
$$= \mathbb{P}(B|A) \cdot \mathbb{P}(A) + \mathbb{P}(B|A^C) \cdot \mathbb{P}(A^C)$$

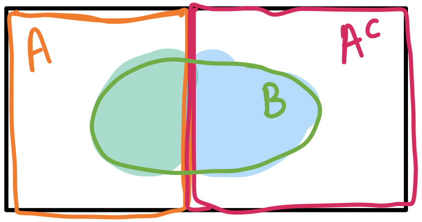
$P(B \cap A) = P(A)P(B|A)$

① Let $A = AFAB$ $A^c = AMAB$
$B =$ colorblind

② $P(B) = ?$

③ order/rep N/A

$P(B|A) = 0.01$
$P(B|A^c) = 0.05$
$P(A) = P(A^c) = 0.5$



④ $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$
$= (0.01)(0.5) + (0.05)(0.5)$
$= 0.03$

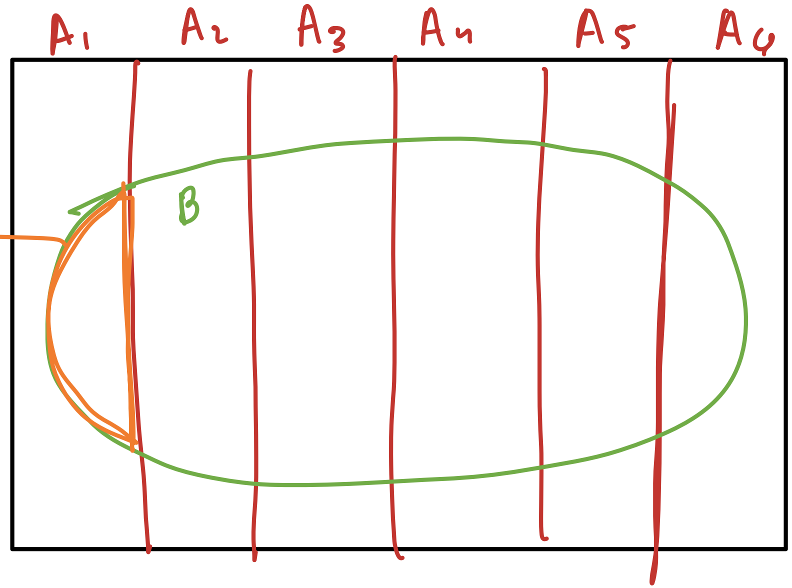
⑤ Probability of choosing random person who is colorblind is 0.03

General Law of Total Proability

Law of Total Probability (general)

If $\{A_i\}_{i=1}^{n} = \{A_1, A_2, \ldots, A_n\}$ form a partition of the sample space, then for event B,

$$\begin{aligned}\mathbb{P}(B) &= \sum_{i=1}^{n} \underline{\mathbb{P}(B \cap A_i)} \\ &= \sum_{i=1}^{n} \underline{\mathbb{P}(B|A_i) \cdot \mathbb{P}(A_i)}\end{aligned}$$



$P(A_1 \cap B)$
$= P(B|A_1)P(A_1)$

Calculating probability with generalized Law of Total Probability

3 events

Example 3

Individuals are diagnosed with a particular type of cancer that can take on three different disease forms,* D_1, D_2, and D_3. It is known that amongst people diagnosed with this particular type of cancer,

$\rightarrow P(D_1)$

- 20% of people will eventually be diagnosed with form D_1,
- 30% with form D_2, and $\rightarrow P(D_2)$
- 50% with form D_3. $\rightarrow P(D_3)$

The probability of requiring chemotherapy (C) differs among the three forms of disease:

- 80% with D_1, $\rightarrow P(C|D_1)$
- 30% with D_2, and $\rightarrow P(C|D_2)$
- 10% with D_3. $\rightarrow P(C|D_3)$

Based solely on the preliminary test of being diagnosed with the cancer, what is the probability of requiring chemotherapy (the event C)?

① Event notation in problem

② $P(C)$?

③ replication and/or order matter? only considering 1 case so N/A

④ labelled parts in problem

Total prob law:

$$P(C) = P(D_1 \cap C) + P(C \cap D_2) + P(C \cap D_3)$$

$$P(C) = P(D_1)P(C|D_1) + P(D_2)P(C|D_2) + P(D_3)P(C|D_3)$$

$$P(C) = 0.2 \cdot 0.8 + 0.3 \cdot 0.3 + 0.5 \cdot 0.1$$

$$= 0.16 + 0.09 + 0.05$$

$$= 0.3$$

⑤ The probability of requiring chemotherapy if you are diagnosed with cancer is 0.3

Let's revisit the color-blind example

Example 4

Recall the color-blind example (Example 2), where

- a person is AMAB with probability 0.5, $P(A^c)$
- AMAB people are color-blind with probability 0.05, and $P(B|A^c)$
- all people are color-blind with probability 0.03. $P(B)$

Assuming people are AMAB or AFAB, find the probability that a color-blind person is AMAB.

① in ex 2

② $P(A^c | B) = ?$

③ N/A

④ $$P(A^c | B) = \frac{P(A^c \cap B)}{P(B)} = \frac{P(B|A^c)\,P(A^c)}{P(B)} = \frac{(0.05)(0.5)}{0.03} = 0.8\overline{33}$$

0.03 → law of total prob

⑤ the probability that a colorblind person is AMAB is $0.8\overline{33}$

Calculate probability with both rules

$P(B \cap A) = P(A|B) P(B)$

Example 5

Suppose

- 1% of people who are AFAB aged 40-50 years have breast cancer, $P(B) = 0.01$
- an AFAB person with breast cancer has a 90% chance of a positive test from a mammogram, and $P(A|B) = 0.9$
- an AFAB person has a 10% chance of a false-positive result from a mammogram. $P(A|B^c) = 0.1$

What is the probability that an AFAB person has breast cancer given that they just had a positive test?

① S = AFAB 40-50 yo
A = positive result mammo
B = breast cancer

② $P(B|A) = ?$

③ N/A

④ $P(B|A) = \dfrac{P(B \cap A)}{P(A)}$ ← Law of total prob

$P(B^c) = 0.99$

$1 - P(B^c) = P(B)$

$$= \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

$P(A \cap B) + P(A \cap B^c)$

$$= \frac{0.9(0.01)}{0.9(0.01) + (0.1)(0.99)}$$

$$= 0.833$$

⑤ The probability of breast cancer given a positive test is 0.833.

Bayes' Rule

Theorem: Bayes' Rule

If $\{A_i\}_{i=1}^{n}$ form a partition of the sample space S, with $\mathbb{P}(A_i) > 0$ for $i = 1 \ldots n$ and $\mathbb{P}(B) > 0$, then

$$\mathbb{P}(A_j|B) = \frac{\mathbb{P}(B|A_j) \cdot \mathbb{P}(A_j)}{\sum_{i=1}^{n} \mathbb{P}(B|A_i) \cdot \mathbb{P}(A_i)}$$