Chapter 8: Probability Mass Functions (pmf's) and Cumulative Distribution Functions (cdf's)

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Learning Objectives

1. Calculate probabilities for discrete random variables
2. Calculate and graph a probability mass function (pmf)
3. Calculate and graph a cumulative distribution function (CDF)

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Probability axioms
- Probability properties
- Counting
- Independence
- Conditional probability
- Bayes' Theorem
- Random Variables

Probability for discrete random variables

- Functions: pmfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Probability for continuous random variables

- Calculus
- Functions: pdfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

What is a probability mass function?

Definition: probability distribution or probability mass function (pmf)

The **probability distribution** or **probability mass function** (**pmf**) of a discrete r.v. X is defined for every number x by

$$\boxed{p_X(x) = \mathbb{P}(X = x)} = \mathbb{P}(\text{all } \omega \in S : X(\omega) = x)$$

Let's demonstrate this definition with our coin toss

p = prob tails
$\underline{1-p}$ = prob heads

Example 1

Suppose we toss 3 coins with probability of tails p. If X is the random variable counting the number of tails, what are the probabilities of each value of X?

$S = \{HHH, HHT, \ldots, TTT\}$

$X(\omega) = x \rightarrow x = 0, 1, 2, 3$

↳ # tails

$P(X=0) = P(HHH) = 1(1-p)^3$

$P(H) \cdot P(H) P(H)$

$P(X=1) = P(HHT \text{ or } HTH \text{ or } THH)$

$= (1-p)(1-p)p + (1-p)p(1-p) + p(1-p)(1-p)$

$= 3p(1-p)^2$

$P(X=2) = P(TTH) + P(THT) + P(HTT)$

$= 3p^2(1-p)$

$P(X=3) = P(TTT) = p^3$

$P(X=x) = \binom{3}{x} p^x (1-p)^{3-x}$

$[P_X(x)]$

$x = 0, 1, 2, 3$

$\binom{3}{0} = 1 \quad \binom{3}{1} = 3$

$\binom{3}{2} = 3 \quad \binom{3}{3} = 1$

Remarks on the pmf

Properties of pmf

A pmf $p_X(x)$ must satisfy the following properties:

- $0 \leq p_X(x) \leq 1$ for all x.
- $\sum_{\{all\ x\}} p_X(x) = 1$.

- Some distributions depend on parameters
  - Each value of a parameter gives a different pmf
  - In previous example, the number of coins tossed was a parameter
    - We tossed 3 coins
    - If we tossed 4 coins, we'd get a different pmf!
  - The collection of all pmf's for different values of the parameters is called a *family* of pmf's

Binomial family of RVs

Example 2

Suppose you toss n coins, each with probability of tails p. If X is the number of tails, what is the pmf of X ?

3 tosses:

$$P_X(x) = \binom{3}{x} p^x (1-p)^{3-x}$$

4 tosses:

$$P_X(x) = \binom{4}{x} p^x (1-p)^{4-x}$$

Binomial PDF :

$$\boxed{P_X(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, 2, 3, \ldots, n}$$

Bernoulli family of RVs

Example 3

Suppose you toss 1 coin, with probability of tails p. If X is the number of tails, what is the pmf of X?

$$p_X(x) = \binom{1}{x} p^x (1-p)^{1-x}$$

$$\binom{1}{0} = 1 \quad \binom{1}{1} = 1$$

$X = 0$

$X = 1$

$$\boxed{p_X(x) = p^x (1-p)^{1-x} \quad \text{for } x = 0, 1}$$

Bernoulli family

$$p_X(x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

Household size (1/5)

Example 4

The table below shows household sizes in 2019. Data are from the U.S. Census.

Size	1	2	3	4	5 or more
Percent	28%	35%	15%	13%	9%

1. What is the sample space for household sizes?
2. Define the random variable for household sizes.
3. Do the values in the table create a pmf? Why or why not?
4. Make a plot of the pmf.
5. Write the cdf as a function.
6. Graph the cdf of household sizes in 2019.

Household size (2/5)

Example 4

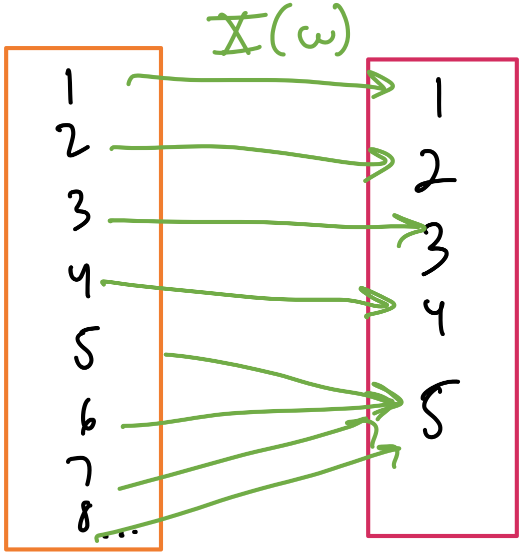
The table below shows household sizes in 2019. Data are from the U.S. Census.

Size	1	2	3	4	5 or more
Percent	28%	35%	15%	13%	9%

1. What is the sample space for household sizes?
2. Define the random variable for household sizes.

① $S = \{1, 2, 3, 4, 5, 6, 7 \ldots\}$

② $X(\omega) = \begin{cases} \omega & \omega = 1, 2, 3, 4 \\ 5+ & \omega = 5, 6, 7, 8, \ldots \end{cases}$



Household size (3/5)

Example 4

The table below shows household sizes in 2019. Data are from the U.S. Census.

Size	1	2	3	4	5 or more
Percent	28%	35%	15%	13%	9%

3. Do the values in the table create a pmf? Why or why not?

4. Make a plot of the pmf

3

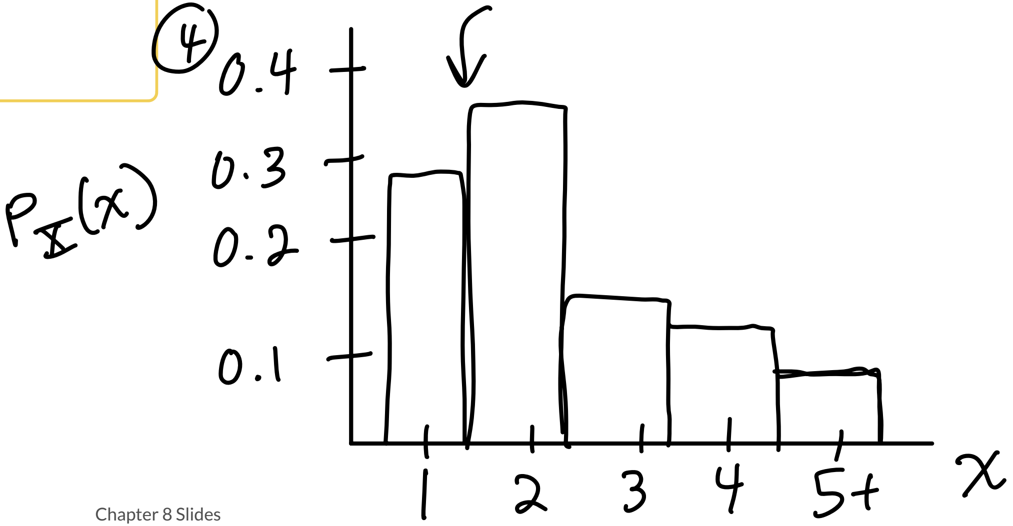
$$0 \le P_X(x) \le 1$$

$$\sum_{\{all\ x\}} P_X(x) = 1\ ?$$

$$0.28 + 0.35 + 0.15 + 0.13 + 0.09 = 1$$

yes! viable pmf

4



What is a cumulative distribution function?

Definition: cumulative distribution function (CDF)

The **cumulative distribution function** (**cdf**) of a discrete r.v. X with pmf $p_X(x)$, is defined for every value x by

$$\underline{F_X(x)} = \underline{\mathbb{P}(X \leq x)} = \sum_{\{all\ y:\ y \leq x\}} \underline{p_X(y)}$$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

coin toss exa

Household size (4/5)

Example 4

The table below shows household sizes in 2019. Data are from the U.S. Census.

Size	1	2	3	4	5 ~~or more~~
Percent	28%	35%	15%	13%	9%

5. Write the cdf as a function.

$$F_X(x) = \sum_{y \le x} p_X(y)$$

$x=$

$$F_X(1) = P(X \le 1) = P(X=1) = 0.28$$

$x=$

$$F_X(2) = P(X \le 2) = P(X=1) + P(X=2) = 0.28 + 0.35 = 0.63$$

$x=$

$$F_X(3) = P(X \le 3) = P(X=1) + P(X=2) + P(X=3) = F_X(2) + P(X=3) = 0.63 + 0.15 = 0.78$$

$$F_X(4) = P(X \le 4) = F_X(3) + P(X=4) = 0.78 + 0.13 = 0.91$$

$$F_X(5) = P(X \le 5) = F_X(4) + P(X=5) = 0.91 + 0.09 = 1$$

$$F_X(x) = \begin{cases} 0 & x < 1 \\ 0.28 & 1 \le x < 2 \\ 0.63 & 2 \le x < 3 \\ 0.78 & 3 \le x < 4 \\ 0.91 & 4 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

$P(X \le x)$

Household size (5/5)

Example 4

The table below shows household sizes in 2019. Data are from the U.S. Census.

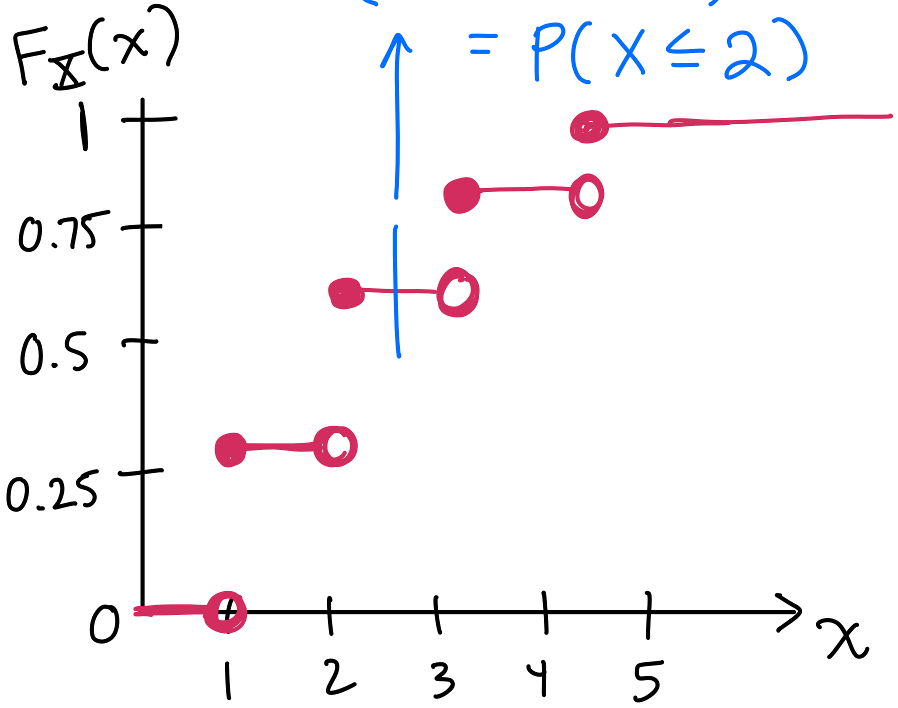
Size	1	2	3	4	5 or more
Percent	28%	35%	15%	13%	9%

6. Graph the cdf of household sizes in 2019.

$$F_X(x) = \begin{cases} 0 & x < 1 \\ 0.28 & 1 \le x < 2 \\ 0.63 & 2 \le x < 3 \\ 0.78 & 3 \le x < 4 \\ 0.91 & 4 \le x < 5 \\ 1 & x \ge 5 \end{cases}$$

$P(X \le x)$

$P(X \le 2.5) = P(X \le 2)$



$F_X(x) = P(X \le x)$

2.5

Properties of *discrete* CDFs

- $F(x)$ is increasing or flat (never decreasing) b/c always adding prob b/w 0 & 1
- $\min_x F(x) = 0$
- $\max_x F(x) = 1$
- CDF is a step function

→ for discrete RVs