

Chapter 9: Independence and Conditioning (Joint Distributions)

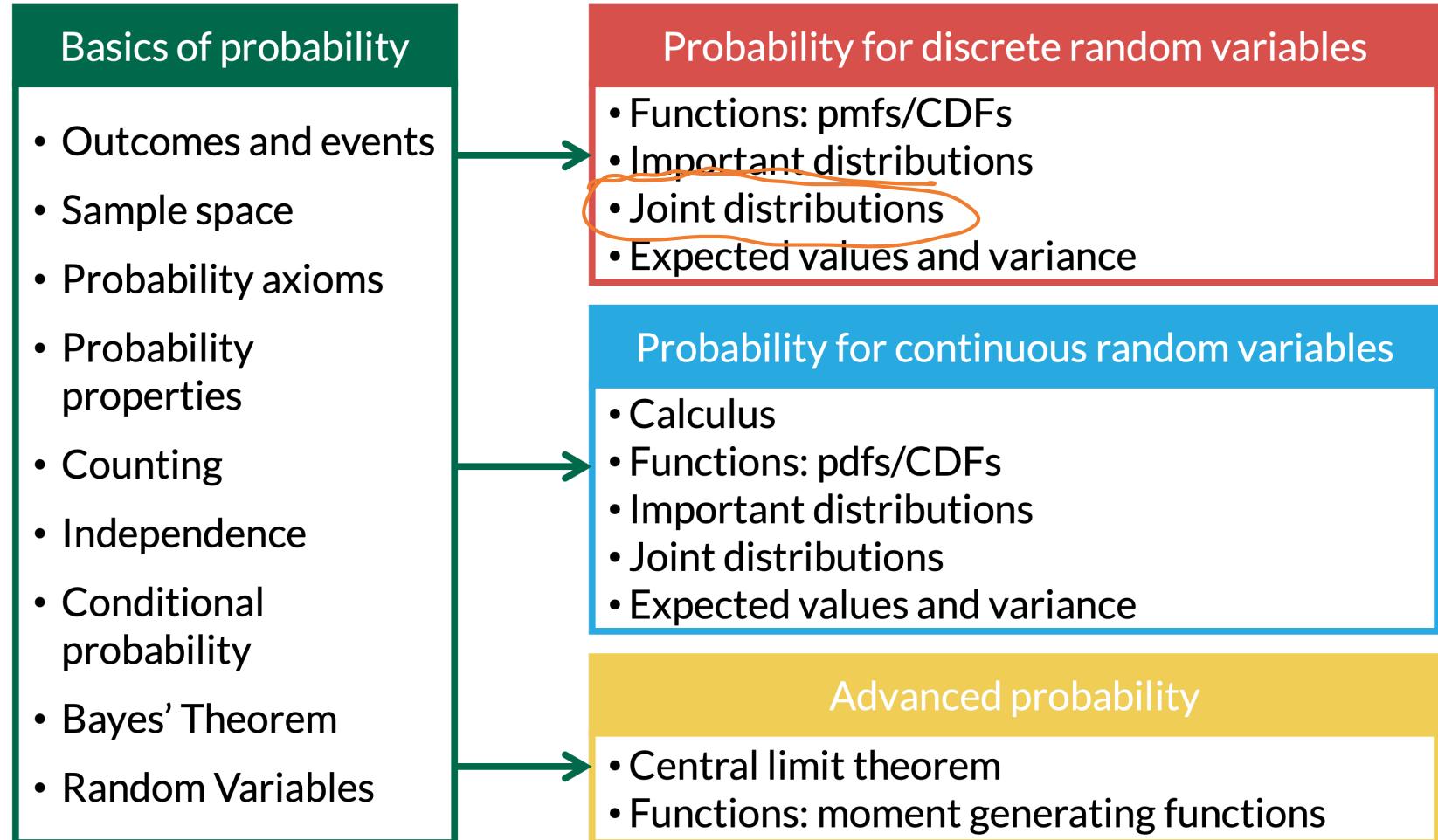
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Learning Objectives

1. Calculate probabilities for a pair of discrete random variables
2. Calculate a joint, marginal, and conditional probability mass function (pmf)
3. Calculate a joint, marginal, and conditional cumulative distribution function (CDF)

Where are we?

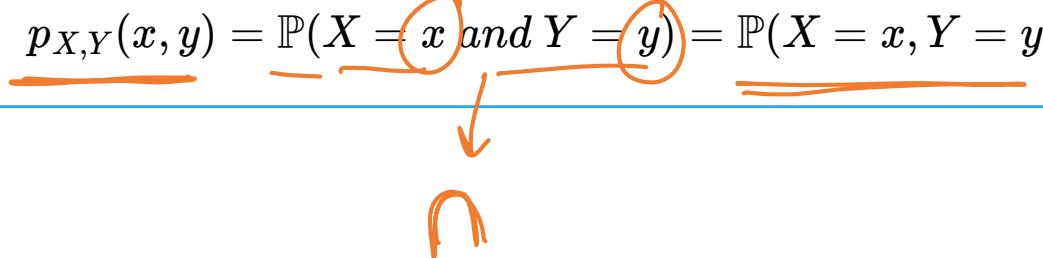


What is a joint pmf?

Definition: joint pmf

The **joint pmf** of a pair of discrete r.v.'s X and Y is

$$p_{X,Y}(x,y) = \mathbb{P}(X = x \text{ and } Y = y) = \mathbb{P}(X = x, Y = y)$$



This chapter's main example

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

1. Find $p_{X,Y}(x,y)$.
2. Find $\mathbb{P}(X + Y = 3)$.
3. Find $\mathbb{P}(Y = 1)$.
4. Find $\mathbb{P}(Y \leq 2)$.
5. Find the joint CDF $F_{X,Y}(x,y)$ for the joint pmf $p_{X,Y}(x,y)$
6. Find the marginal CDFs $F_X(x)$ and $F_Y(y)$
7. Find $p_{X|Y}(x|y)$.
8. Are X and Y independent? Why or why not?

Joint pmf

intersection

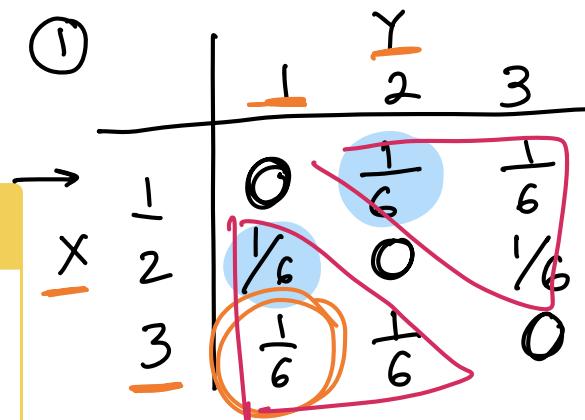
Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

- Find $p_{X,Y}(x,y)$. ✓
- Find $\mathbb{P}(X + Y = 3)$.

$$\textcircled{2} \quad P(X + Y = 3) ?$$

$$X + Y = 3$$



$$P(X=x, Y=y) = \begin{cases} \frac{1}{6} & x \neq y \\ 0 & x = y \end{cases}$$

for $x = 1, 2, 3$
 $y = 1, 2, 3$

joint pmf
for X & Y

$$\begin{aligned} P(X + Y = 3) &= P(X=2, Y=1) + P(X=1, Y=2) \\ &= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} X = 1, Y = 1 \\ \hookrightarrow \text{impossible} \end{aligned}$$

$$\begin{aligned} X = 1, Y = 2 \\ \hookrightarrow P(X \& Y) &= P(X)P(Y|X) \\ &= \frac{1}{3} \left(\frac{1}{2}\right) = \frac{1}{6} \end{aligned}$$

Marginal pmf's

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

3. Find $\underline{\underline{P(Y = 1)}}$.

4. Find $\underline{\underline{P(Y \leq 2)}}$.

		Y	1	2	3	$P_X(x)$
	X	1	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
	2	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$	
	3	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{3}$	
$p_Y(y) =$		$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$		1

$$\textcircled{3} \quad \underline{\underline{P(Y = 1)}} = P(X = 1, Y = 1) + P(X = 2, Y = 1) + P(X = 3, Y = 1)$$

$$= \sum_{y=1}^1 \sum_{x=1}^3 P_{X,Y}(x,y) = 0 + \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

$$\textcircled{4} \quad \underline{\underline{P(Y \leq 2)}} = P(Y = 1) + P(Y = 2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \rightarrow \text{sum from marginal}$$

$$= \sum_{y=1}^2 \sum_{x=1}^3 P_{X,Y}(x,y) \leftarrow \text{sum directly from joint}$$

Remarks on the joint pmf

Some properties of joint pmf's:

- A joint pmf $p_{X,Y}(x,y)$ must satisfy the following properties:

- $p_{X,Y}(x,y) \geq 0$ for all x, y .
- $\sum_{\{all\} x} \sum_{\{all\} y} p_{X,Y}(x,y) = 1.$

- Marginal pmf's:

- $p_X(x) = \sum_{\{all\} y} p_{X,Y}(x,y)$

- $p_Y(y) = \sum_{\{all\} x} p_{X,Y}(x,y)$

What is a joint CDF?

Definition: joint CDF

The **joint CDF** of a pair of discrete r.v.'s X and Y is



$$F_{X,Y}(x,y) = \mathbb{P}(\underline{X \leq x} \text{ and } \underline{Y \leq y}) = \mathbb{P}(X \leq x, Y \leq y)$$

Joint CDFs

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

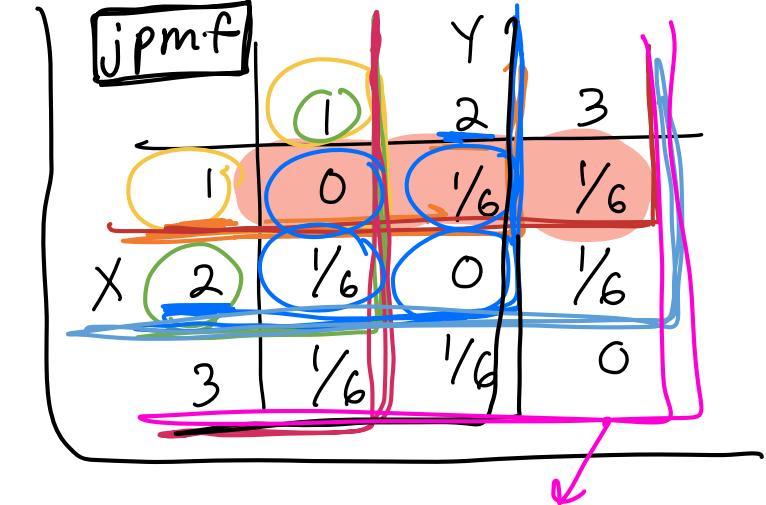
5. Find the joint CDF $F_{X,Y}(x,y)$ for the joint pmf $p_{X,Y}(x,y)$

$$P(X \leq x \text{ & } Y \leq y)$$

$$P(X \leq 1, Y \leq 1) = P(X = 1, Y = 1)$$

$$\begin{aligned} P(X \leq 3, Y \leq 2) &= P(X = 3, Y = 1) + P(X = 3, Y = 2) + \\ &\quad P(X = 2, Y = 1) + P(X = 2, Y = 2) + \\ &\quad P(X = 1, Y = 1) + P(X = 1, Y = 2) \\ &= \sum_{x=1}^3 \sum_{y=1}^2 P_{X,Y}(x,y) \end{aligned}$$

		Y		
		1	2	3
X	1	0	$\frac{1}{6}$	$\frac{1}{3}$
	2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$
	3	$\frac{1}{3}$	$\frac{2}{3}$	1



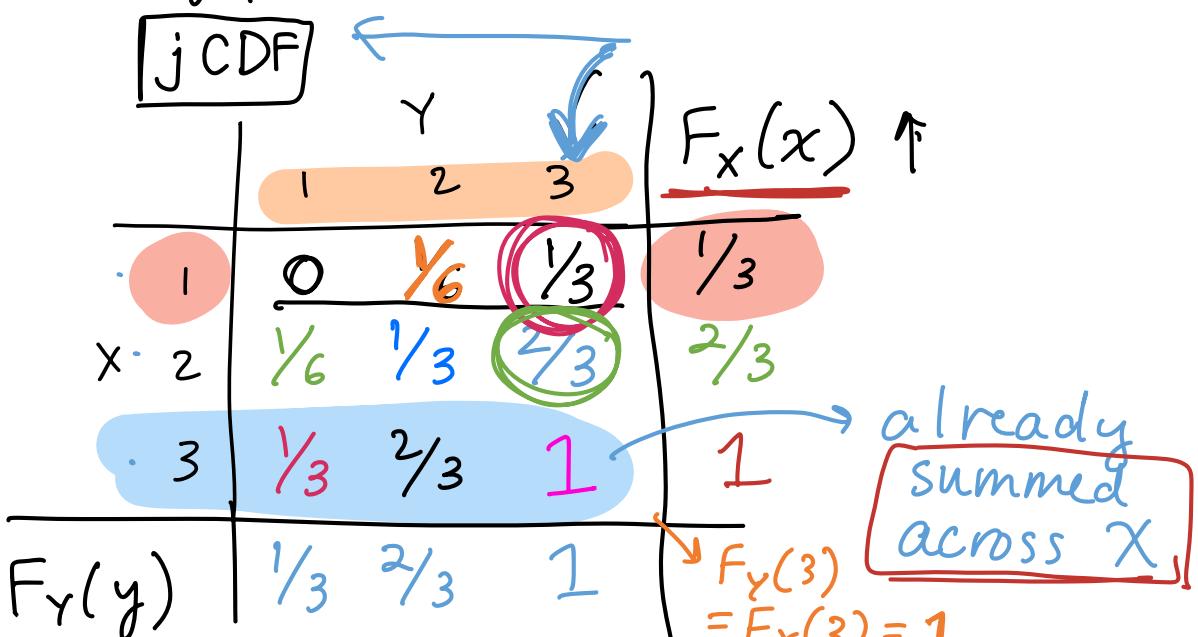
$$\sum_{\text{all } x} \sum_{\text{all } y} P_{X,Y}(x,y) = 1$$

Marginal CDFs

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

6. Find the marginal CDFs $F_X(x)$ and $F_Y(y)$



$$\textcircled{6} \quad F_X(1) = P(X \leq 1) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$$

$$F_X(2) = P(X \leq 2) = P(X=1) + P(X=2)$$

for marginal y , sum across all x
 for marginal x , sum across all y

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{3} & 1 \leq x < 2 \\ \frac{2}{3} & 2 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$$

Remarks on the joint and marginal CDF

- $\underline{F_X(x)}$: right most columns of the CDf table (where the \underline{Y} values are largest)
- $\underline{F_Y(y)}$: bottom row of the table (where \underline{X} values are largest)
- $F_X(x) = \lim_{\substack{y \rightarrow \infty}} F_{X,Y}(x, y)$
- $F_Y(y) = \lim_{x \rightarrow \infty} F_{X,Y}(x, y)$

Independence and Conditioning

Recall that for events A and B ,

- $\underbrace{\mathbb{P}(A|B)}_{\mathbb{P}(A \cap B) / \mathbb{P}(B)}$
- A and B are independent if and only if
 - $\mathbb{P}(A|B) = \underbrace{\mathbb{P}(A)}_{\mathbb{P}(A \cap B) / \mathbb{P}(B)}$
 - $\underbrace{\mathbb{P}(A \cap B)}_{\mathbb{P}(A) \cdot \mathbb{P}(B)} = \underbrace{\mathbb{P}(A)}_{\mathbb{P}(A \cap B) / \mathbb{P}(B)} \cdot \underbrace{\mathbb{P}(B)}_{\mathbb{P}(A \cap B) / \mathbb{P}(B)}$

Independence and conditioning are defined similarly for r.v.'s, since

$$\frac{p_X(x) = \mathbb{P}(X = x)}{\mathbb{P}(X)} \text{ and } \frac{p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)}{\mathbb{P}(X \cap Y)}.$$

if X & Y are independent:

$$P(X=x, Y=y) = \underbrace{P(X=x)P(Y=y)}_{\mathbb{P}(X \cap Y)}$$

What is the conditional pmf?

Definition: conditional pmf

The **conditional pmf** of a pair of discrete r.v.'s X and Y is defined as

$$p_{X|Y}(x|y) = \underbrace{\mathbb{P}(X = x | Y = y)}_{\text{if } p_Y(y) > 0} = \frac{\mathbb{P}(X = x \text{ and } Y = y)}{\mathbb{P}(Y = y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)} > 0$$

joint
marginal

if $p_Y(y) > 0$.

$$\text{if Ind: } P_{X|Y}(x|y) = P(X=x)$$

$$\text{when } p_Y(y) = 0, \quad P_{X|Y}(x|y) = 0$$

Remarks on the conditional pmf

The following properties follow from the conditional pmf definition:

- If $X \perp Y$ (independent)

- $p_{X|Y}(x|y) = p_X(x)$ for all x and y
- $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for all x and y
- Which also implies (\Rightarrow): $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ for all x and y

- If X_1, X_2, \dots, X_n are independent

- $p_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) = P(X_1 = x_1, X_2 = x_2, \dots, X_n = x_n) = \prod_{i=1}^n p_{X_i}(x_i)$
- $F_{X_1, X_2, \dots, X_n}(x_1, x_2, \dots, x_n) = P(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) = \prod_{i=1}^n P(X_i \leq x_i) = \prod_{i=1}^n \underline{F_{X_i}(x_i)}$

$$P_{X_1}(x_1) P_{X_2}(x_2) \cdots P_{X_n}(x_n)$$

$$\prod_{i=1}^n p_{X_i}(x_i)$$

$$\prod_{i=1}^n \underline{F_{X_i}(x_i)}$$

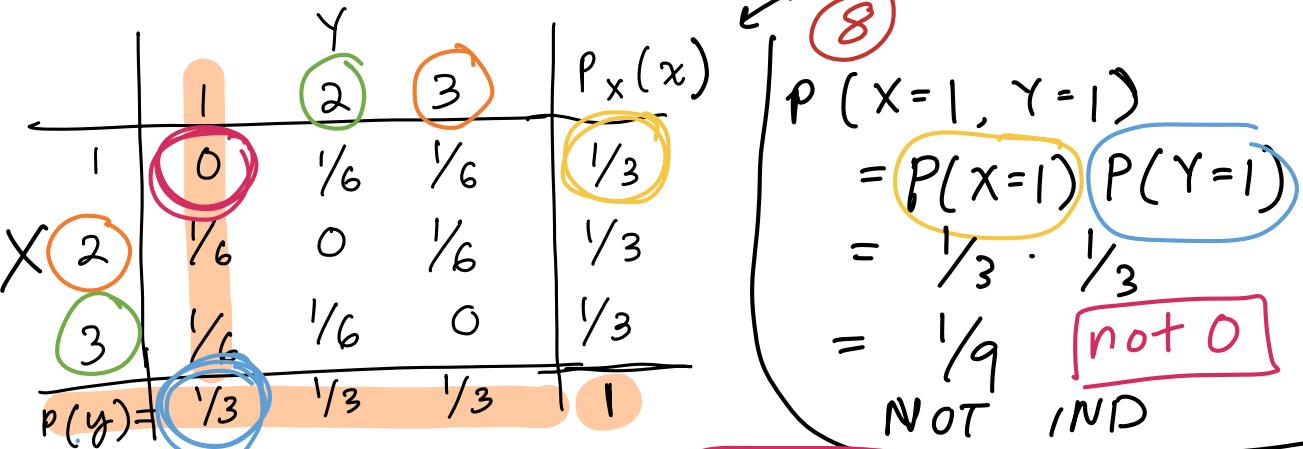
Conditional pmf's

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

7. Find $p_{X|Y}(x|y)$.

8. Are X and Y independent? Why or why not?



$$⑦ P_{X|Y}(X=1 | Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{0}{\frac{1}{3}} = 0$$

↳ ex of diagonal in table

Remark:

- To show that X and Y are not independent, we just need to find one counter example
- However, to show that they are independent, we need to verify this for all possible pairs of x and y

2 ex of off diag:

$$P(X=2 | Y=3) = \frac{P(X=2, Y=3)}{P(Y=3)}$$

$$P(X=3 | Y=2) = \frac{P(X=3, Y=2)}{P(Y=2)}$$

$$= \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2}$$

$P_{X|Y}(x|y) = \begin{cases} \frac{1}{2} & x \neq y \\ 0 & x = y \end{cases}$

for $x, y = 1, 2, 3$

