Chapter 9: Independence and Conditioning (Joint Distributions)

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Learning Objectives

1. Calculate probabilities for a pair of discrete random variables
2. Calculate a *joint*, *marginal*, *and conditional* probability mass function (pmf)
3. Calculate a *joint*, *marginal*, *and conditional* cumulative distribution function (CDF)

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Probability axioms
- Probability properties
- Counting
- Independence
- Conditional probability
- Bayes' Theorem
- Random Variables

Probability for discrete random variables

- Functions: pmfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Probability for continuous random variables

- Calculus
- Functions: pdfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

What is a joint pmf?

Definition: joint pmf

The **joint pmf** of a pair of discrete r.v.'s X and Y is

$$p_{X,Y}(x,y) = \mathbb{P}(X = x \; and \; Y = y) = \mathbb{P}(X = x, Y = y)$$

$\cap$

This chapter's main example

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

1. Find $p_{X,Y}(x, y)$.
2. Find $\mathbb{P}(X + Y = 3)$.
3. Find $\mathbb{P}(Y = 1)$.
4. Find $\mathbb{P}(Y \leq 2)$.
5. Find the joint CDF $F_{X,Y}(x, y)$ for the joint pmf $p_{X,Y}(x, y)$
6. Find the marginal CDFs $F_X(x)$ and $F_Y(y)$
7. Find $p_{X|Y}(x|y)$.
8. Are X and Y independent? Why or why not?

Joint pmf

intersection

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

1. Find $p_{X,Y}(x,y)$. ✓
2. Find $\mathbb{P}(X+Y=3)$.

①

$X \backslash Y$	1	2	3
1	0	$\frac{1}{6}$	$\frac{1}{6}$
2	$\frac{1}{6}$	0	$\frac{1}{6}$
3	$\frac{1}{6}$	$\frac{1}{6}$	0

$X=1,\ Y=1$

↳ impossible

$X=1,\ Y=2$

$2|1$

↳ $P(X \,\&\, Y) = P(X)P(Y|X)$

$= \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6}$

$$P(X=x, Y=y) = \begin{cases} \frac{1}{6} & x \neq y \\ 0 & x = y \end{cases}$$

for $x = 1, 2, 3$

$y = 1, 2, 3$

joint pmf for X & Y

② $P(X+Y=3)$?

$X+Y=3$

$$P(X+Y=3) = P(X=2, Y=1) + P(X=1, Y=2)$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$$

Marginal pmf's

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

3. Find $\mathbb{P}(Y = 1)$.

4. Find $\mathbb{P}(Y \leq 2)$.

X \ Y	1	2	3	$p_X(x)$
1	0	1/6	1/6	1/3
2	1/6	0	1/6	1/3
3	1/6	1/6	0	1/3
$p_Y(y)=$	1/3	1/3	1/3	1

(3) $P(Y=1) = P(X=1, Y=1) + P(X=2, Y=1) + P(X=3, Y=1)$

$$= \sum_{y=1}^{1} \sum_{x=1}^{3} p_{X,Y}(x,y) = 0 + \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

(4) $P(Y \leq 2) = P(Y=1) + P(Y=2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$ → sum from marginal

$$= \sum_{y=1}^{2} \sum_{x=1}^{3} p_{X,Y}(x,y)$$ ← sum directly from joint

Remarks on the joint pmf

Some properties of joint pmf's:

- A joint pmf $p_{X,Y}(x,y)$ must satisfy the following properties:
  - $p_{X,Y}(x,y) \geq 0$ for all x, y.
  - $\sum_{\{all\ x\}} \sum_{\{all\ y\}} p_{X,Y}(x,y) = 1$.
- Marginal pmf's:
  - $p_X(x) = \sum_{\{all\ y\}} p_{X,Y}(x,y)$
  - $p_Y(y) = \sum_{\{all\ x\}} p_{X,Y}(x,y)$

What is a joint CDF?

Definition: joint CDF

The **joint CDF** of a pair of discrete r.v.'s X and Y is

$\leq$

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x \; and \; Y \leq y) = \mathbb{P}(X \leq x, Y \leq y)$$

Joint CDFs

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

5. Find the joint CDF $F_{X,Y}(x,y)$ for the joint pmf $p_{X,Y}(x,y)$

jCDF

X \ Y	1	2	3
1	0	1/6	1/3
2	1/6	1/3	2/3
3	1/3	2/3	1

jpmf

X \ Y	1	2	3
1	0	1/6	1/6
2	1/6	0	1/6
3	1/6	1/6	0

$$\sum_{\text{all } x} \sum_{\text{all } y} p_{X,Y}(x,y) = 1$$

$P(X \le x \ \& \ Y \le y)$

$P(X \le 1, Y \le 1) = P(X = 1, Y = 1)$

$P(X \le 3, Y \le 2) = P(X=3, Y=1) + P(X=3, Y=2) + P(X=2, Y=1) + P(X=2, Y=2) + P(X=1, Y=1) + P(X=1, Y=2)$

$$= \sum_{x=1}^{3} \sum_{y=1}^{2} p_{X,Y}(x,y)$$

Marginal CDFs

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

6. Find the marginal CDFs $F_X(x)$ and $F_Y(y)$

jCDF

X \ Y	1	2	3	$F_X(x)$
1	0	1/6	1/3	1/3
2	1/6	1/3	2/3	2/3
3	1/3	2/3	1	1
$F_Y(y)$	1/3	2/3	1	

already summed across X

$F_Y(3) = F_X(3) = 1$

⑥ $F_X(1) = P(X \le 1) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$

$F_X(2) = P(X \le 2) = P(X=1) + P(X=2)$

for marginal y, sum across all x
marginal x, sum across all y

$$F_X(x) = \begin{cases} 0 & x < 1 \\ 1/3 & 1 \le x < 2 \\ 2/3 & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

Remarks on the joint and marginal CDF

- $F_X(x)$: right most columns of the CDf table (where the Y values are largest)
- $F_Y(y)$: bottom row of the table (where X values are largest)
- $F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y)$
- $F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x,y)$

Independence and Conditioning

Recall that for *events* A and B,

- $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$
- A and B are independent if and only if
  - $\mathbb{P}(A|B) = \mathbb{P}(A)$
  - $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

Independence and conditioning are defined similarly for r.v.'s, since

$$\underbrace{p_X(x) = \mathbb{P}(X = x)}_{p(X)} \text{ and } \underbrace{p_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y)}_{p(X \cap Y)}.$$

if X & Y are independent:

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

What is the conditional pmf?

Definition: conditional pmf

The **conditional pmf** of a pair of discrete r.v.'s X and Y is defined as

joint

$$p_{X|Y}(x|y) = \mathbb{P}(X = x | Y = y) = \frac{\mathbb{P}(X = x \textit{ and } Y = y)}{\mathbb{P}(Y = y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)} > 0$$

marginal

if $p_Y(y) > 0$.

if ind: $p_{X|Y}(x|y) = P(X = x)$

when $p_Y(y) = 0$, $p_{X|Y}(x|y) = 0$

Remarks on the conditional pmf

The following properties follow from the conditional pmf definition:

- If $X \perp Y$ (independent)
  - $p_{X|Y}(x|y) = p_X(x)$ for all x and y
  - $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for all x and y
  - Which also implies ($\Rightarrow$): $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ for all x and y
- If $X_1, X_2, \ldots, X_n$ are independent
  - $$p_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{i=1}^{n} p_{X_i}(x_i)$$
  - $$F_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n) = \prod_{i=1}^{n} P(X_i \le x_i) = \prod_{i=1}^{n} F_{X_i}(x_i)$$

$p_{X_1}(x_1) p_{X_2}(x_2) \cdot \ldots p_{X_n}(x_n)$

Conditional pmf's

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

7. Find $p_{X|Y}(x|y)$.
8. Are X and Y independent? Why or why not?

Remark:

- To show that X and Y are *not* independent, we just need to find one counter example
- However, to show that they *are* independent, we need to verify this for all possible pairs of x and y

$X \backslash Y$	1	2	3	$p_X(x)$
1	0	1/6	1/6	1/3
2	1/6	0	1/6	1/3
3	1/6	1/6	0	1/3
$p(y)=$	1/3	1/3	1/3	1

8

$P(X=1, Y=1)$

$= P(X=1)\,P(Y=1)$

$= 1/3 \cdot 1/3$

$= 1/9$ $\quad$ $\boxed{\text{not } 0}$

NOT IND

7 $\quad P_{X|Y}(X=1 \mid Y=1) = \dfrac{P(X=1, Y=1)}{P(Y=1)} = \dfrac{0}{1/3} = 0$

$P(Y=1) \to 1/3$

↳ ex of diagonal in table

2 ex of off diag:

$P(X=3 \mid Y=2) = \dfrac{P(X=3, Y=2)}{P(Y=2)} = \dfrac{1/6}{1/3} = \dfrac{1}{2}$

$P(X=2 \mid Y=3) = \dfrac{P(X=2, Y=3)}{P(Y=3)} = \dfrac{1/6}{1/3} = \dfrac{1}{2}$

$$P_{X|Y}(x|y) = \begin{cases} 1/2 & x \neq y \\ 0 & x = y \end{cases}$$

for $x, y = 1, 2, 3$