Chapter 10: Expected Values of Discrete RVs

Meike Niederhausen and Nicky Wakim

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Learning Objectives

1. Calculate the mean (expected value) of discrete random variables

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Probability axioms
- Probability properties
- Counting
- Independence
- Conditional probability
- Bayes' Theorem
- Random Variables

Probability for discrete random variables

- Functions: pmfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Probability for continuous random variables

- Calculus
- Functions: pdfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

Our good and fair friend, the 6-sided die

Example 1

Suppose you roll a fair 6-sided die. What value do you expect to get?

$$\text{avg} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6}$$

$$= 3.5$$

$$\text{weighted avg} = \underbrace{\left(\frac{1}{6}\right)}_{P(X=1)}1 + \underbrace{\left(\frac{1}{6}\right)}_{P(X=2)}2 + \left(\frac{1}{6}\right)3 + \left(\frac{1}{6}\right)4 + \left(\frac{1}{6}\right)5 + \left(\frac{1}{6}\right)6$$

$$= 3.5$$

What is an expected value?

Definition: Expected value

The **expected value** of a discrete r.v. X that takes on values $x_1, x_2, \ldots, x_n$ is

$$\mathbb{E}[X] = \sum_{i=1}^{n} x_i p_X(x_i).$$

- Expected values are not necessarily an actual outcome
  - In previous example, we cannot roll a 3.5
  - It could be that our expected value is not in the sample space ($E(X) \notin S$)
- Definition holds when X takes on countably infinitely many values:

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} x_i p_X(x_i)$$

Our good and not-so-fair friend, the 6-sided die

$p_X(x_i)$

Example 2

Suppose the die is 6-sided, but not fair. And the probabilities of each side is distributed as:

x	$p_X(x)$
1	0.10
2	0.05
3	0.02
4	0.30
5	0.50
6	0.03

What value do you expect to get on a roll?

$$E(X) = \sum_{i=1}^{6} x_i \underline{P(X = x_i)}$$

$$= 1(0.10) + 2(0.05) + 3(0.02) + 4(0.3) + 5(0.5) + 6(\underline{0.03})$$

$$= 4.14$$

- still weighted avg
- do NOT expected value to nearest whole #

Expected value of a Bernoulli distribution

Example 3

Suppose

$$X = \begin{cases} 1 & \text{with probability } p \quad \text{(success)} \\ 0 & \text{with probability } 1-p \quad \text{(failure)} \end{cases}$$

Find the expected value of X.

$$E(X) = \sum_{i=1}^{n} x_i P(X = x_i) = \underline{0(1-p)} + 1(p)$$

$$= p$$

Let's slightly change our random variable

Example 5

Suppose

$$X = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

Find the expected value of X.

$$E(X) = \sum_{i=1}^{2} x_i P(X = x_i) = (1)(p) + (-1)(1-p)$$

$$= p - (1-p) = p \cancel{+} - 1 + p$$

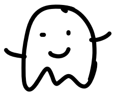
$$= 2p - 1$$

$p = \frac{1}{2}$: $2p - 1 = 2(\frac{1}{2}) - 1 = 0$

$p > \frac{1}{2}$: $2p - 1 > 0$ $\quad 2(\frac{2}{3}) - 1 = \frac{4}{3} - 1 = \frac{1}{3}$

$p < \frac{1}{2}$: $2p - 1 < 0$

$2(\frac{1}{3}) - 1 = \frac{2}{3} - 1 = -\frac{1}{3}$

Ghost! 

Example 6

A ghost is trick-or-treating. It comes to a house where it is known that there are 30 candies in the bag and only one is a watermelon Jolly Rancher, which is the ghost's favorite. The ghost takes pieces of candy without replacement until it gets the watermelon Jolly Rancher. How many pieces of candy do we expect the ghost to take?

Can we model this with a distribution?

let X = # draws to get WJR

what is $p_X(x_i)$

★ show how to do w/ counting

$P(X=1) = \frac{1}{30} \rightarrow$

$P(X=2) = \frac{\cancel{29}}{30} \cdot \frac{1}{\cancel{29}} = \frac{1}{30} \longrightarrow$

$P(X=3) = \frac{\cancel{29}}{30} \cdot \frac{\cancel{28}}{\cancel{29}} \cdot \frac{1}{\cancel{28}} = \frac{1}{30}$

$P(X=2) = P(\text{1st not WJR})$

if ind, $\frac{1}{30}$

$\cdot\, P(\text{2nd WJR} \mid \text{1st NOT WJR})$

$P(\text{1st NOT} \cap \text{2nd is})$

$P(A \cap B \cap C) = P(A)P(B)P(C)$

when $A \not\perp B \not\perp C \longrightarrow$

$P(A \cap B \cap C)$

$= P(A)\,P(B, C \mid A) = P(A)\,P(B \mid A)\,P(C \mid B, A)$

$$P(X = j) = \frac{1}{30} \text{ for } j = 1, 2, 3, \ldots, 30$$

$$E(X) = \sum_{i=1}^{30} x_i P(X = x_i) = \sum_{i=1}^{30} x_i \left(\frac{1}{30}\right)$$

$$= \frac{1}{30} \sum_{i=1}^{30} x_i = \frac{1}{30}(1 + 2 + 3 + \ldots + 30)$$

$$= \frac{465}{30} = 15.5$$

We expect the ghost to take 15.5 candies until it get the WJR (including the WJR)