Chapter 22: Introduction to Counting

Meike Niederhausen and Nicky Wakim

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Learning Objectives

1. Define permutations and combinations
2. Characterize difference between sampling with and without replacement
3. Characterize difference between sampling when order matters and when order does not matter
4. Calculate the probability of sampling any combination of the following: *with or without replacement* and *order does or does not matter*

Where are we?

Basics of probability

- Outcomes and events
- Sample space
- Probability axioms
- Probability properties
- Counting
- Independence
- Conditional probability
- Bayes' Theorem
- Random Variables

Probability for discrete random variables

- Functions: pmfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Probability for continuous random variables

- Calculus
- Functions: pdfs/CDFs
- Important distributions
- Joint distributions
- Expected values and variance

Advanced probability

- Central limit theorem
- Functions: moment generating functions

Basic Counting Examples

Basic Counting Examples (1/3)

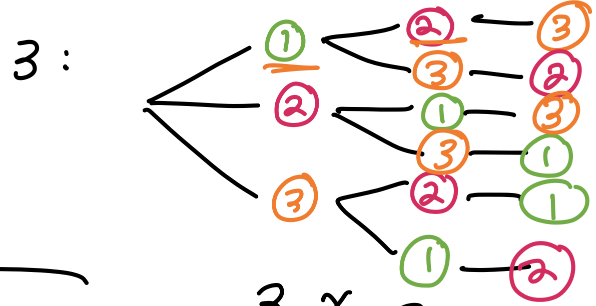
Example 1

Suppose we have 10 (distinguishable) subjects for study.

1. How many possible ways are there to order them?
2. How many ways to order them if we can reuse the same subject and
   - need 10 total?
   - need 6 total?
3. How many ways to order them *without replacement* and only need 6?
4. How many ways to choose 6 subjects without replacement if the order doesn't matter?

Basic Counting Examples (2/3)

Suppose we have 10 (distinguishable) subjects for study.



$3 \times 2 \times 1 = 3!$ factorial

Example 1.1

How many possible ways are there to order them?

not replacing, order matters

$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 10!$

Example 1.2

How many ways to order them if we can reuse the same subject and

replacement

- need 10 total?
- need 6 total?

$10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^{10}$

$10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$

Basic Counting Examples (3/3)

Suppose we have 10 (distinguishable) subjects for study.

Example 1.3

How many ways to order them without replacement and only need 6?

$10 \times 9 \times 8 \times 7 \times 6 \times 5 = \frac{10!}{4!}$ $\neq 10!$

$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1}$

Example 1.4

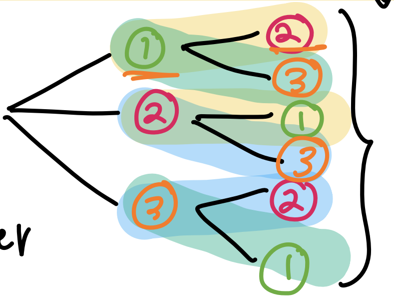
How many ways to choose 6 subjects without replacement if the order doesn't matter?

$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5}{6!} = \frac{\left(\frac{10!}{4!}\right)}{6!} = \frac{10!}{4!\,6!}$

How many ways to order 6 subj? ex 1.1

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

3 subj, want 2, order does NOT matter



$\frac{3 \cdot 2}{2}$ picking 2 w/ order / # times duplicated

$\frac{\left(\frac{3!}{1!}\right)}{2!}$

Permutations and Combinations

Permutations and Combinations

Definition: Permutations w/out replacement ex 1.3 $0! = 1$

Permutations are the number of ways to **arrange in order** r distinct objects when there are n total.

$$nPr = \frac{n!}{(n-r)!}$$

6 10

ex 1.1: $n=10$, $r=10$ $\frac{10!}{(10-10)!} = 10!$

Definition: Combinations

Combinations are the number of ways to choose (**order doesn't matter**) r objects from n without replacement.

$$nCr = \text{"n choose r"} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

ex 1.4 $n=10$
$r=6$
$n-r = 10-6$

Some combinations properties

Property	Proof
n choose R $\binom{n}{r} = \binom{n}{n-r}$	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!}$ r = n-r r n - r
n ch 1 $\binom{n}{1} = n$	$\binom{n}{1} = \frac{n!}{1!(n-1)!} = \frac{n\cdot(n-1)\cdot(n-2)\cdots 1}{1!\cdot(n-1)\cdot(n-2)\cdots 1} = \frac{n\cdot(n-1)!}{1\cdot(n-1)!} = \frac{n}{1} = n$
n ch 0 $\binom{n}{0} = 1$	$\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{1\cdot n!} = 1$

More Examples: order matters vs. not

Table of different cases

See table on pg. 277 of textbook

- n = total number of objects
- r = number objects needed

	with replacement	**without replacement**	
order matters	n^r	$nPr = \dfrac{n!}{(n-r)!}$	perm's
order doesn't matter	$\binom{n+r-1}{r}$	$nCr = \binom{n}{r} = \dfrac{n!}{r!(n-r)!}$	combos

Enumerating Events and Sample Space

- Recall, $P(A) = \frac{|A|}{|S|}$
  - Within combinatorics, we can use the previous equations to help enumerate the event and sample space
  - But A might be a combination of enumerations

- For example in the following example drawing 2 spades when order does not matter, we actually need to enumerate the other cards that are NOT spades. So the event is choosing 2 spades out of 13 AND choosing 0 other cards of 39 cards (13 hearts + 13 clubs + 13 diamonds).
- Thus the probability is actually:

$$P(\text{two spades}) = \frac{\binom{13}{2}\binom{39}{0}}{\binom{52}{2}}$$

- Note that $13 + 39 = 52$ and $2 + 0 = 2$. So the numerator's n's add up to the denominator's n and the numerator's r's add up to the denominator's r's

More examples: order matters vs. not (1/2)

Example 2

Suppose we draw 2 cards from a standard deck without replacement. What is the probability that both are spades when

1. order matters?
2. order doesn't matter?

Let A = both cards spades (and no other suits)

S = picking 2 cards

① S: $52 \cdot 51 = {}_{52}P_2 = \frac{52!}{(52-2)!} = 52 \cdot 51$ ($\frac{52!}{50!} \to 52 \cdot 51 \cdot 50!$)

A: $\underbrace{13 \cdot 12}_{\text{picking 2 spades}} \times {}_{39}P_0 = \frac{39!}{(39-0)!} = 1$ ($13 \cdot 12 = {}_{13}P_2$)

$$P(A) = \frac{|A|}{|S|} = \frac{13 \cdot 12}{52 \cdot 51} = \left(\frac{13}{52}\right) \cdot \left(\frac{12}{51}\right)$$

② S: ${}_{52}C_2 = \frac{52!}{2!(52-2)!}$ (n = 52, r = 2)

A: ${}_{13}C_2 \cdot {}_{39}C_0$, $\frac{13!}{2!(13-2)!}$, $\frac{39!}{0!(39-0)!} = 1$

$$P(A) = \frac{\binom{13}{2}}{\binom{52}{2}} = \frac{\frac{13!}{2!\,11!}}{\frac{52!}{2!\,50!}} = \frac{13 \cdot 12}{52 \cdot 51}$$

("13C2", "52C2")