Chapter 24: Continuous RVs and PDFs

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Learning Objectives

1. Distinguish between discrete and continuous random variables.
2. Calculate probabilities for continuous random variables.
3. Calculate ~~and graph~~ a density (i.e., probability density function, PDF).
4. Calculate ~~and graph a~~ CDF (i.e., a cumulative distribution function)

Discrete vs. Continuous RVs

- For a **discrete** RV, the set of possible values is either finite or can be put into a countably infinite list.

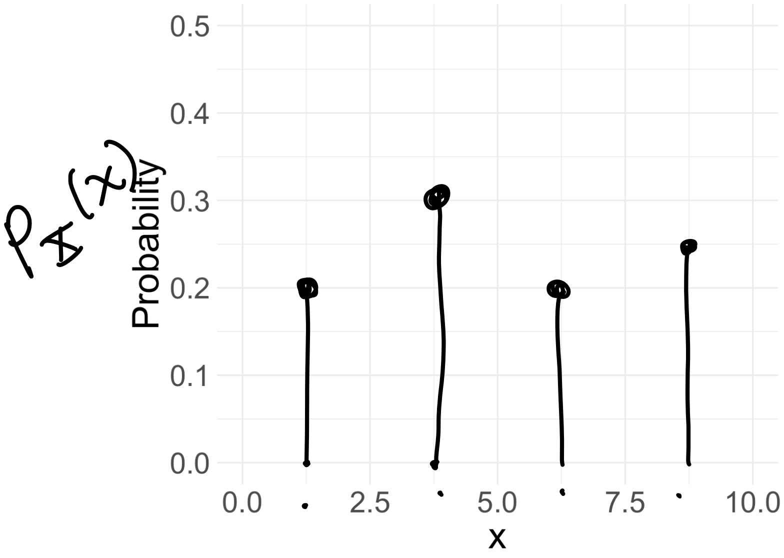
- **Continuous** RVs take on values from continuous *intervals*, or unions of continuous intervals

	Discrete	Continuous
probability function	mass (probability mass function; PMF)	density (probability density function; PDF)
	$0 \le p_X(x) \le 1$	$0 \le f_X(x)$ (not necessarily ≤ 1)
	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$
	$P(0 \le X \le 2)$ $= P(X=0) + P(X=1) + P(X=2)$ if X is integer valued	$P(0 \le X \le 2)$ $= \int_0^2 f_X(x)\,dx$
	$P(X \le 3) \ne P(X < 3)$ when $P(X=3) \ne 0$	$P(X \le 3) = P(X < 3)$ since $P(X=3) = 0$ always
cumulative distribution function (CDF) $F_X(x)$	$F_X(a) = P(X \le a)$ $= \sum_{x \le a} P(X=a)$ graph of CDF is a step function with jumps of the same size as the mass, from 0 to 1	$F_X(a) = P(X \le a)$ $= \int_{-\infty}^{a} f_X(x)\,dx$ graph of CDF is nonnegative and continuous, rising up from 0 to 1
examples	counting: defects, hits, die values, coin heads/tails, people, card arrangements, trials until success, etc.	lifetimes, waiting times, height, weight, length, proportions, areas, volumes, physical quantities, etc.
named distributions	Bernoulli, Binomial, Geometric, Negative Binomial, Poisson, Hypergeometric, Discrete Uniform	Continuous Uniform, Exponential, Gamma, Beta, Normal
expected value	$\mathbb{E}(X) = \sum_x x p_X(x)$ $\mathbb{E}(g(X)) = \sum_x g(x) p_X(x)$	$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$ $\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$
$\mathbb{E}(X^2)$	$\mathbb{E}(X^2) = \sum_x x^2 p_X(x)$	$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$
variance	$\text{Var}(X) =$ $\mathbb{E}(X^2) - (\mathbb{E}(X))^2$	$\text{Var}(X) =$ $\mathbb{E}(X^2) - (\mathbb{E}(X))^2$
std. dev.	$\sigma_X = \sqrt{\text{Var}(X)}$	$\sigma_X = \sqrt{\text{Var}(X)}$

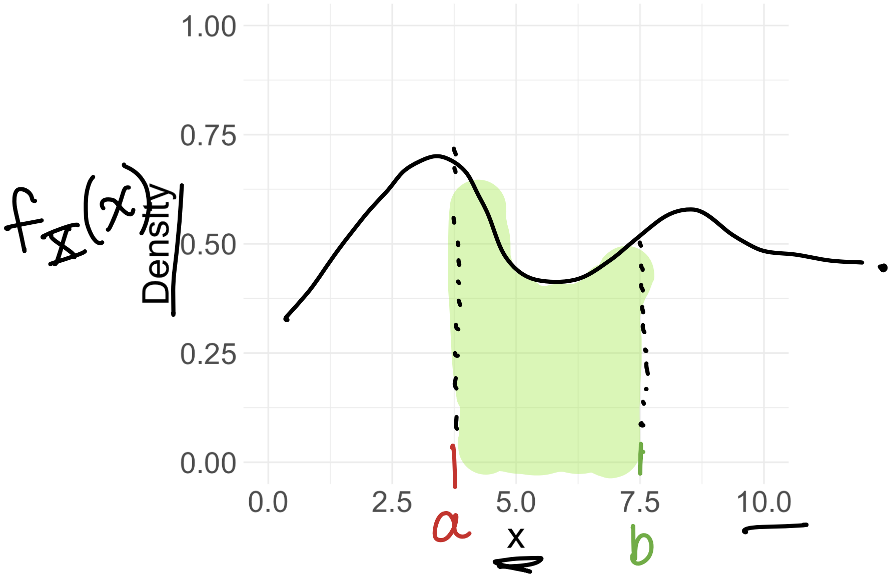
Figure from Introduction to Probability TB (pg. 301)

How to define probabilities for continuous RVs?

Discrete RV X:



Continuous RV X:



- pmf: $p_X(x) = P(X = x)$
- density: $f_X(x)$ $\neq P(X = x)$
- probability: $P(a \leq X \leq b) = \int_a^b f_X(x)dx$

What is a probability density function?

Probability density function

The probability distribution, or **probability density function (pdf)**, of a continuous random variable X is a function $f_X(x)$, such that for all real values a, b with $a \leq b$,

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$$

Remarks:

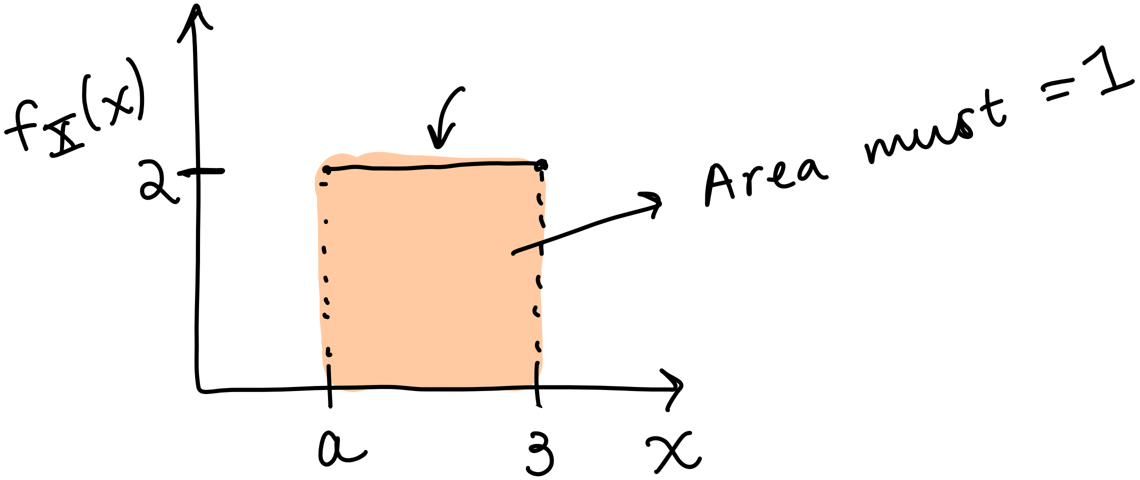
1. Note that $f_X(x) \neq \mathbb{P}(X = x)$!!!
2. In order for $f_X(x)$ to be a pdf, it needs to satisfy the properties
   - $f_X(x) \geq 0$ for all x
   - $\int_{-\infty}^{\infty} f_X(x)dx = 1$

Let's demonstrate the PDF with an example (1/5)

Example 1.1

Let $f_X(x) = 2$, for $a \leq x \leq 3$.

1. Find the value of a so that $f_X(x)$ is a pdf.



geom : area of rect $h \cdot w = 2 \cdot (3-a)$

$$2(3-a) = 1$$
$$6 - 2a = 1$$
$$5 = 2a \rightarrow a = 2.5$$

OR $\int_a^3 f_X(x)\,dx = 1$

$$\int_a^3 2\,dx = 2x \Big|_{x=a}^{x=3}$$
$$= 2(3) - 2a$$
$$\Rightarrow 6 - 2a = 1$$
$$a = 2.5$$

Let's demonstrate the PDF with an example (2/5)

Example 1.2

Let $f_X(x) = 2$, for $\underset{2.5}{a} \le x \le 3$.

2. Find $\mathbb{P}(2.7 \le X \le 2.9)$.

$$P(2.7 \le X \le 2.9) = \int_{2.7}^{2.9} 2 \, dx$$

check integrands are w/in bounds of $f(x)$

$\cancel{3.5}$ 3

$$= 2x \Big|_{x=2.7}^{x=2.9} = 2(2.9) - 2(2.7)$$

$$= 0.4$$

$$P(2.7 \le X \le 2.9) = 0.4$$

Let's demonstrate the PDF with an example (3/5)

Example 1.3

Let $f_X(x) = 2$, for $a \leq x \leq 3$.

3. Find $\mathbb{P}(2.7 < X \leq 2.9)$.

$\leq$

in last problem

$P(X = x) = 0$

so

$P(a \leq X \leq b) = P(a < X < b)$

same as ex 1.2:

$\boxed{0.4}$

Let's demonstrate the PDF with an example (4/5)

Example 1.4

Let $f_X(x) = 2$, for $a \leq x \leq 3$.

4. Find $\mathbb{P}(X = 2.9)$.

$$P(X = 2.9) = \int_{2.9}^{2.9} f_X(x)\,dx$$

$$= \int_{2.9}^{2.9} 2\,dx$$

$$= 2x \Big|_{x=2.9}^{x=2.9}$$

$$= 2(2.9 - 2.9) = 0$$

RULE:

$$\boxed{P(X = x) = 0}$$

Let's demonstrate the PDF with an example (5/5)

Example 1.5

Let $f_X(x) = 2$, for $a \leq x \leq 3$.

2.5

5. Find $\mathbb{P}(X \leq 2.8)$.

$$P(X \leq 2.8) = \int_{2.5}^{2.8} 2\, dx$$

$$= 2x \Big|_{2.5}^{2.8}$$

$$= 2(2.8 - 2.5)$$

$$= 0.6$$

$F_X(x)$

how CDF is defined

What is a cumulative distribution function?

Cumulative distribution function

The **cumulative distribution function (cdf)** of a continuous random variable X, is the function $F_X(x)$, such that for all real values of x,

$$F_X(x) = \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f_X(s)\,ds$$

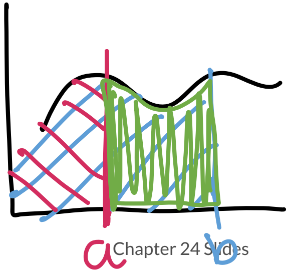
s is dummy variable

if $f_X(x)$ is bound so $a \leq x \leq b$

$$\int_a^x f_X(s)\,ds$$

Remarks: In general, $F_X(x)$ is increasing and

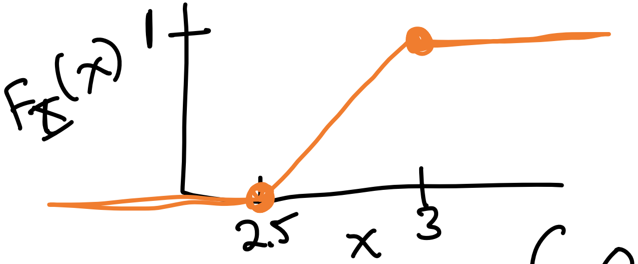
- $\lim_{x\to-\infty} F_X(x) = 0$
- $\lim_{x\to\infty} F_X(x) = 1$
- $P(X > a) = 1 - P(X \leq a) = 1 - F_X(a)$
- $P(a \leq X \leq b) = F_X(b) - F_X(a)$



Let's demonstrate the CDF with an example

Example 2

Let $f_X(x) = 2$, for $2.5 \leq x \leq 3$. Find $F_X(x)$.



$$F_X(x) = P(X \leq x) = \int_{2.5}^{x} f_X(s)\,ds$$

$$= \int_{2.5}^{x} 2\,ds$$

$$= 2s \Big|_{s=2.5}^{s=x}$$

$$= 2x - 2(2.5)$$

$$= 2x - 5$$

$$F_X(x) = \begin{cases} 0 & x > 2.5 \\ 2x-5 & 2.5 \leq x \leq 3 \\ 1 & x > 3 \end{cases}$$

$x = 3$: $F_X(3) = 2(3) - 5 = 6 - 5 = 1$

Derivatives of the CDF

Theorem 1

If X is a continuous random variable with pdf $f_X(x)$ and cdf $F_X(x)$, then for all real values of x at which $F'_X(x)$ exists,

$$\frac{d}{dx} F_X(x) = F'_X(x) = f_X(x)$$

Finding the PDF from a CDF

Example 3

Let X be a RV with cdf

$$F_X(x) = \begin{cases} 0 & x < 2.5 \\ 2x - 5 & 2.5 \leq x \leq 3 \\ 1 & x > 3 \end{cases}$$

Find the pdf $f_X(x)$.

$$f(x) = F'(x) = \begin{cases} 0 & x < 2.5 \\ 2 & 2.5 \leq x \leq 3 \\ 0 & x > 3 \end{cases}$$

$f_X(x) = 2$ for $2.5 \leq x \leq 3$ ✓

Let's go through another example (1/7)

Example 4

Let X be a RV with pdf $f_X(x) = 2e^{-2x}$, for $x > 0$.

1. Show $f_X(x)$ is a pdf.
2. Find $\mathbb{P}(1 \leq X \leq 3)$.
3. Find $F_X(x)$.
4. Given $F_X(x)$, find $f_X(x)$.
5. Find $\mathbb{P}(X \geq 1 | X \leq 3)$.
6. Find the median of the distribution of X.

Let's go through another example (2/7)

Example 4.1

Let X be a RV with pdf $f_X(x) = 2e^{-2x}$, for $x > 0$.

1. Show $f_X(x)$ is a pdf.

Show

① $f_X(x) > 0$ for all x ✓

② $\int_{-\infty}^{\infty} f_X(x) = 1$ ✓

$\Rightarrow f_X(x)$ is a pdf!

① $f(x) = 2\underbrace{e^{-2x}} > 0$ ✓

$x = 1$ $\quad \frac{1}{e^{-2}}$ $\qquad x = 1000$ $\quad \frac{1}{e^{-1000}}$

② $\int_0^\infty 2e^{\overset{u}{-2x}} dx$

$u = -2x$

$du = -2dx$

$2dx = -du$

bounds: $u = -2x$

$u = -2(\infty) = -\infty$

$u = -2(0) = 0$

$\int_0^{-\infty} e^u (-du) = (-)(-)\int_{-\infty}^0 e^u du$

$(-)$ $\quad +$

$= e^u \Big|_{u=-\infty}^{u=0} = e^0 - e^{-\infty}$ ✓

$= 1 - 0 = 1$

Let's go through another example (3/7)

Do this problem at home for extra practice.

Example 4.2

Let X be a RV with pdf $f_X(x) = 2e^{-2x}$, for $x > 0$.

2. Find $\mathbb{P}(1 \leq X \leq 3)$.

$$P(1 \leq X \leq 3) = \int_1^3 2e^{-2x}\,dx$$

u-sub in 4.1

$$= -e^{-2x}\Big|_1^3$$

$$= -e^{-2(3)} - (-e^{-2(1)})$$

$$= -e^{-6} + e^{-2}$$

$$\boxed{P(1 \leq X \leq 3) = e^{-2} - e^{-6}}$$

Let's go through another example (4/7)

Example 4.3

Let X be a RV with pdf $f_X(x) = 2e^{-2x}$, for $x \geq 0$.

3. Find $F_X(x)$.

$$F_X(x) = P(X \leq x) = \int_0^x f_X(s)\,ds$$

$$= \int_0^x 2e^{-2s}\,ds$$

$$= -e^{-2s}\Big|_0^x = -e^{-2x} + (+e^{-2(0)})$$

$$= -e^{-2x} + 1 = 1 - e^{-2x}$$

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-2x} & x > 0 \end{cases}$$

Where is $F_X(x) = 1$? $\lim_{x \to \infty} 1 - \underbrace{e^{-2x}}_{\to 0} = 1$

Let's go through another example (5/7)

Do this problem at home for extra practice.

Example 4.4

Let X be a RV with pdf $f_X(x) = 2e^{-2x}$, for $x > 0$.

4. Given $F_X(x)$, find $f_X(x)$.

$$f_X(x) = \frac{d}{dx} F_X(x) = F'_X(x)$$

$$= \frac{d}{dx}\left(1 - e^{-2x}\right) = 0 - (-2)e^{-2x}$$

$$= 2e^{-2x}$$

$$\boxed{f_X(x) = 2e^{-2x} \quad \text{for } x > 0}$$

Let's go through another example (6/7)

Recall

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Example 4.5

Let X be a RV with pdf $f_X(x) = 2e^{-2x}$, for $x > 0$.

5. Find $\mathbb{P}(X \geq 1 \mid X \leq 3)$.

$$P(X \geq 1 \mid X \leq 3)$$

$$= \frac{P(1 \leq X \leq 3)}{P(X \leq 3)} \rightarrow F_X(3) - F_X(1)$$

bc $P(a \leq X \leq b)$
$= F(a) - F(b)$

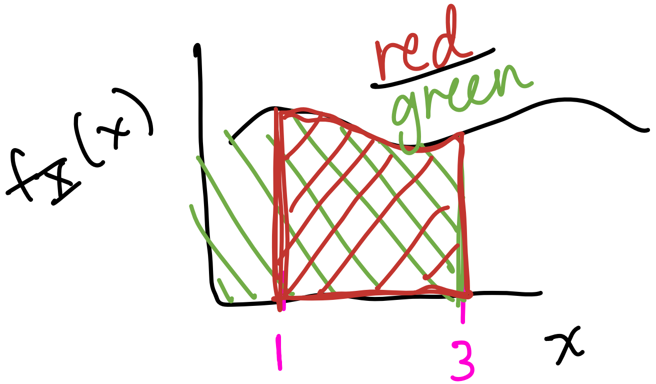
$$F_X(x) = 1 - e^{-2x}$$

$x > 0$

$$= \frac{F_X(3) - F_X(1)}{F_X(3)}$$

$$= \frac{(\cancel{1} - e^{-2(3)}) \cancel{-} (\cancel{1} + e^{-2(1)})}{1 - e^{-2(3)}}$$

$$= \boxed{\frac{e^{-2} - e^{-6}}{1 - e^{-6}}}$$

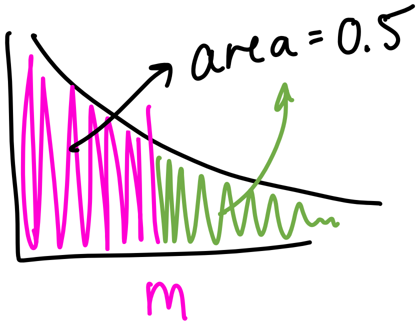


Let's go through another example (7/7)

Example 4.6

Let X be a RV with pdf $f_X(x) = 2e^{-2x}$, for $x > 0$.

6. Find the median of the distribution of X.



median: $P(X \le m) = 0.5$

$$\int_0^m 2e^{-2x}\,dx = 0.5$$

$$\hookrightarrow F_X(m) = 0.5$$

$$1 - e^{-2(m)} = 0.5$$

$$0.5 = e^{-2m}$$

log = ln

$$\log(0.5) = \log\left(e^{-2m}\right)$$

$$\log(0.5) = -2m$$

$$m = \frac{-\log(0.5)}{2} = \boxed{\text{median is } 0.34657}$$