Chapter 25: Joint densities

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Learning Objectives

1. Solve double integrals in our mini lesson!
2. Calculate probabilities for a pair of continuous random variables
3. Calculate a *joint and marginal* probability density function (pdf)
4. Calculate a *joint and marginal* cumulative distribution function (CDF) from a pdf

Double Integrals Mini Lesson (1/3)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Mini Lesson Example 1

Solve the following integral:

$$\int_2^3 \int_0^1 xy\,dy\,dx$$

Double Integrals Mini Lesson (2/3)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Mini Lesson Example 2

Solve the following integral:

$$\int_2^3 \int_0^1 (x + y) dy dx$$

Double Integrals Mini Lesson (3/3)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Mini Lesson Example 3

Solve the following integral:

$$\int_{2}^{3}\int_{0}^{1} e^{x+y}\,dy\,dx$$

How to define the joint pdf for continuous RVs?

For a single continuous RV X is a function $f_X(x)$, such that for all real values a, b with $a \leq b$,

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$$

For two continuous RVs (X and Y), we can define the **joint pdf**, $f_{X,Y}(x, y)$, such that for all real values a, b, c, d with $a \leq b$ and $c \leq d$,

$$\mathbb{P}(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x, y)dydx$$

Important properties of the joint pdf

1. Note that $f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$!!!
2. In order for $f_{X,Y}(x,y)$ to be a pdf, it needs to satisfy the properties
   - $f_{X,Y}(x,y) \geq 0$ for all x, y
   - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$

What is the joint cumulative distribution function?

Definition: Joint cumulative distribution function (Joint CDF)

The **joint cumulative distribution function (cdf)** of continuous random variables X and Y, is the function $F_{X,Y}(x, y)$, such that for all real values of x and y,

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s, t) dt ds$$

Remarks:

- The definition above for $F_{X,Y}(x, y)$ is a **function** of x and y.
- The joint cdf at the point (a, b), is

$$F_{X,Y}(a, b) = \mathbb{P}(X \leq a, Y \leq b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(s, t) dt ds$$

What are the marginal pdf's?

Definition: Marginal pdf's

Suppose X and Y are continuous r.v.'s, with joint pdf $f_{X,Y}(x, y)$. Then the **marginal probability density functions** are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx$$

Common steps for solving problems

1. Set up the domain of the pdf with a picture
2. Translate to needed integrands
    - For probability: shade in the area of interest, then translate
    - For expected value: translate domain
3. Set up integral: $dxdy$ or $dydx$?
4. Solve integral!

Example of joint pdf

Example 1.1

Let $f_{X,Y}(x,y) = \frac{3}{2}y^2$, for $0 \leq x \leq 2,\ 0 \leq y \leq 1.$

1. Find $\mathbb{P}(0 \leq X \leq 1, 0 \leq Y \leq \frac{1}{2})$.

Example of joint pdf

Example 1.2

Let $f_{X,Y}(x,y) = \frac{3}{2}y^2$, for $0 \le x \le 2,\ 0 \le y \le 1$.

2. Find $f_X(x)$ and $f_Y(y)$.

Example of a *more complicated* joint pdf

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Example 2.1

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

1. Find $f_X(x)$ and $f_Y(y)$.

Example of a *more complicated* joint pdf

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Example 2.2

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

2. Find $\mathbb{P}(Y < 3)$.

Let's complicate this even more!

Example 3.1

Let X and Y have constant density on the square $0 \leq X \leq 4, 0 \leq Y \leq 4$.

1. Find $\mathbb{P}(|X - Y| < 2)$

Finding the pdf of a transformation

- Let M be a transformation of X and Y
- When we have a transformation of X and Y, M, we need to follow a specific process to find the pdf of M

We follow this process:

1. Start with the joint pdf for X and Y
   - aka $f_{X,Y}(x, y)$
2. Translate the domain of X and Y to M
3. Find the CDF of M
   - aka $F_M(m)$ or $P(M \leq m)$
4. Take the derivative of the CDF of M to find the pdf of M
   - aka $f_M(m) = \frac{d}{dm} F_M(m)$

Let's complicate this even more!

Example 3.2

Let X and Y have constant density on the square $0 \leq X \leq 4, 0 \leq Y \leq 4$.

2. Let $M = \max(X, Y)$. Find the pdf for M, that is $f_M(m)$.

Let's complicate this even more!

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Example 3.3

Let X and Y have constant density on the square $0 \le X \le 4, 0 \le Y \le 4$.

3. Let $Z = \min(X, Y)$. Find the pdf for Z, that is $f_Z(z)$.

Let's complicate this even further!

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Example 4

Let X and Y have joint density $f_{X,Y}(x, y) = \frac{8}{5}(x + y)$ in the region $0 < x < 1, \ \frac{1}{2} < y < 1$. Find the pdf of the r.v. Z, where $Z = XY$.