Chapter 25: Joint densities

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Learning Objectives

1. Solve double integrals in our mini lesson!
2. Calculate probabilities for a pair of continuous random variables
3. Calculate a *joint and marginal* probability density function (pdf)
4. Calculate a *joint and marginal* cumulative distribution function (CDF) from a pdf

Double Integrals Mini Lesson (1/3)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Mini Lesson Example 1

Solve the following integral:

$$\int_2^3 \int_0^1 xydydx$$

Double Integrals Mini Lesson (2/3)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Mini Lesson Example 2

Solve the following integral:

$$\int_2^3 \int_0^1 (x+y)dydx$$

Double Integrals Mini Lesson (3/3)

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Mini Lesson Example 3

Solve the following integral:

$$\int_2^3 \int_0^1 e^{x+y} dydx$$

How to define the joint pdf for continuous RVs?

For a single continuous RV X is a function $f_X(x)$, such that for all real values a, b with $a \leq b$,

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx$$

For two continuous RVs (X and Y), we can define the **joint pdf**, $f_{X,Y}(x, y)$, such that for all real values a, b, c, d with $a \leq b$ and $c \leq d$,

$$\mathbb{P}(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d \underbrace{f_{X,Y}(x, y)}_{\text{joint pdf}}dydx$$

Important properties of the joint pdf

1. Note that $f_{X,Y}(x, y) \neq \mathbb{P}(X = x, Y = y)$!!!

2. In order for $f_{X,Y}(x, y)$ to be a pdf, it needs to satisfy the properties

   - $f_{X,Y}(x, y) \geq 0$ for all x, y
   - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx dy = 1$

Sum over entire pdf = 1

What is the joint cumulative distribution function?

Definition: Joint cumulative distribution function (Joint CDF)

The **joint cumulative distribution function (cdf)** of continuous random variables X and Y, is the function $F_{X,Y}(x, y)$, such that for all real values of x and y,

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s, t)dtds$$

Remarks:

- The definition above for $F_{X,Y}(x, y)$ is a **function** of x and y.
- The joint cdf at the point (a, b), is

$$F_{X,Y}(a, b) = \mathbb{P}(X \leq a, Y \leq b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(s, t)dtds$$

lower
bounds of X & Y

What are the marginal pdf's?

Definition: Marginal pdf's

Suppose X and Y are continuous r.v.'s, with joint pdf $f_{X,Y}(x, y)$. Then the **marginal probability density functions** are

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$

integrate over all y

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx$$

integrate over all x

Common steps for solving problems

1. Set up the domain of the joint pdf with a picture
2. Translate to needed integrands
    - For probability: shade in the area of interest, then translate
    - For expected value: translate domain
3. Set up integral: $dxdy$ or $dydx$?
4. Solve integral!

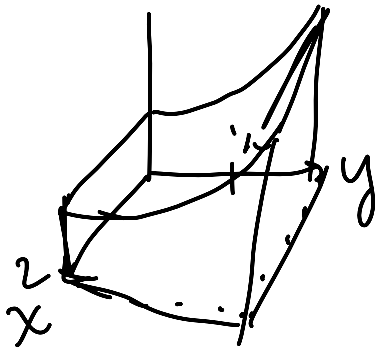
$\int_{\bigcirc}^{\bigcirc}$ integrands

Example of joint pdf

Example 1.1

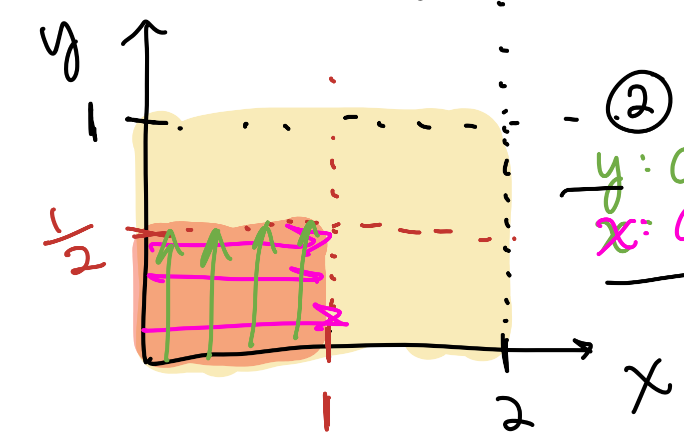
Let $f_{X,Y}(x,y) = \frac{3}{2}y^2$, for $0 \le x \le 2,\ 0 \le y \le 1$.

1. Find $\mathbb{P}(0 \le X \le 1, 0 \le Y \le \frac{1}{2})$.



$$P(0 \le X \le 1, 0 \le Y \le \tfrac{1}{2}) = \frac{1}{16}$$

① Domain x & y



② $y: 0 \to 1/2$

$x: 0 \to 1$

③ ④ $$P(0 \le X \le 1, 0 \le Y \le 1/2) = \int_0^{1/2}\int_0^1 f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^{1/2}\int_0^1 \frac{3}{2}y^2\,dx\,dy$$

$$= \int_0^{1/2}\left[\frac{3}{2}y^2 x\right]_{x=0}^{x=1} dy$$

$$= \int_0^{1/2}\left[\frac{3}{2}y^2(1-0)\right]dy = \int_0^{1/2}\frac{3}{2}y^2\,dy$$

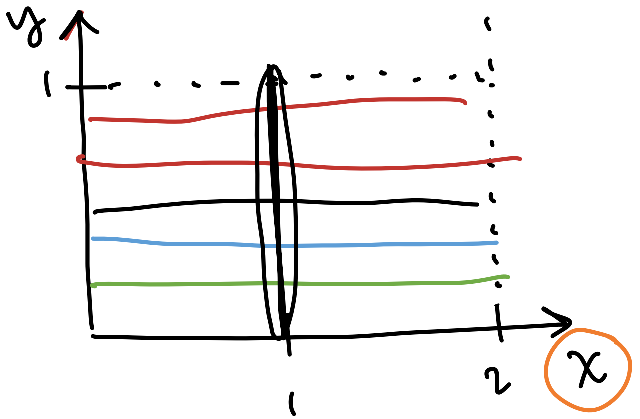
$$= \frac{1}{2}y^3\Big|_0^{1/2} = \frac{1}{2}\left(\frac{1}{2}\right)^3 - \frac{1}{2}(0)^3 = \boxed{\frac{1}{16}}$$

Example of joint pdf

Example 1.2

Let $f_{X,Y}(x,y) = \frac{3}{2}y^2$, for $0 \le x \le 2,\ 0 \le y \le 1$.

2. Find $f_X(x)$ and $f_Y(y)$.



$f(x)$: integrate out y

$$f_X(x) = \int_0^1 f_{X,Y}(x,y)\,dy = \int_0^1 \frac{3}{2}y^2\,dy$$

$$= \frac{1}{2}y^3\Big|_{y=0}^{y=1} = \frac{1}{2}1^3 - \frac{1}{2}0^3 = \frac{1}{2}$$

$$f_X(x) = \frac{1}{2} \quad \text{for} \quad 0 \le x \le 2$$

$f_Y(y)$: int out x

$$f_Y(y) = \int_0^2 \frac{3}{2}y^2\,dx = \frac{3}{2}y^2 x\Big|_{x=0}^{x=2}$$

$$= \frac{3}{2}y^2(2) - \frac{3}{2}y^2(0) = 3y^2$$

$$f_Y(y) = 3y^2 \quad \text{for} \quad 0 \le y \le 1$$

Example of a *more complicated* joint pdf

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Example 2.1

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

1. Find $f_X(x)$ and $f_Y(y)$.

Example of a *more complicated* joint pdf

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Example 2.2

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

2. Find $\mathbb{P}(Y < 3)$.

Let's complicate this even more!

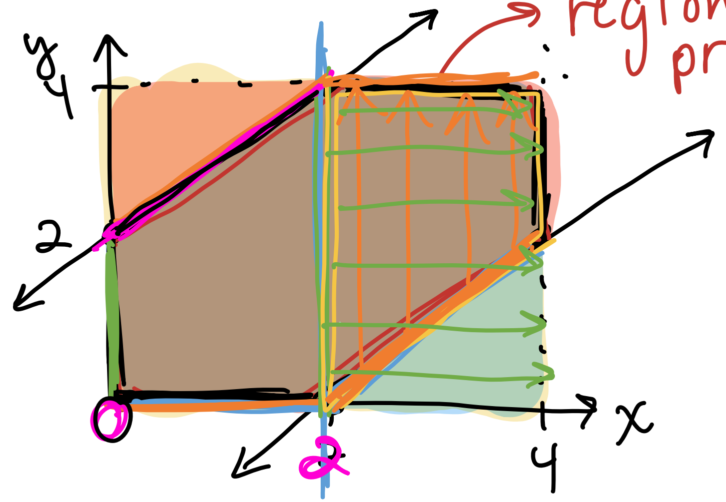
$f_{X,Y}(x,y) = \frac{1}{16}$ for $0 \le x < 4,\ 0 \le y \le 4$

Example 3.1

Let X and Y have constant density on the square $0 \le X \le 4, 0 \le Y \le 4$.

1. Find $\mathbb{P}(|X - Y| < 2)$

① Domain:



② $|X - Y| < 2$

$-2 < x - y < 2$

$-2 < x - y$, $+y$, $+2$: $y < x + 2$

$x - y < 2$: $y > x - 2$

$$P(|X-Y|<2)$$

$$= \int_0^2 \int_0^{x+2} \frac{1}{16}\, dy\, dx + \int_2^4 \int_{x-2}^{4} \frac{1}{16}\, dy\, dx$$

$$= \int_0^2 \frac{1}{16} y \Big|_{y=0}^{y=x+2} dx + \int_2^4 \frac{1}{16} y \Big|_{y=x-2}^{4} dx$$

$$= \int_0^2 \frac{1}{16}x + \frac{1}{8}\, dx + \int_2^4 \frac{1}{4} - \frac{1}{16}x + \frac{1}{8}\, dx$$

$$\frac{3}{8} - \frac{1}{16}x$$

$$= \frac{1}{32}x^2 + \frac{1}{8}x \Big|_0^2 + \frac{3}{8}x - \frac{1}{32}x^2 \Big|_2^4$$

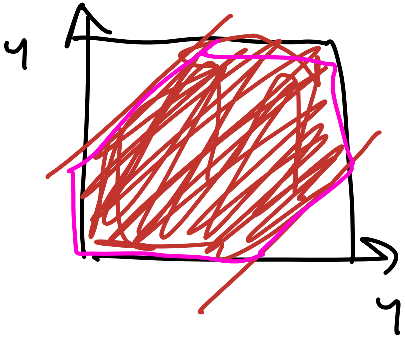
$$= \frac{1}{32}(4) + \frac{2}{8} + \left[\frac{12}{8} - \frac{16}{32} - \left(\frac{6}{8} - \frac{4}{32}\right)\right]$$

$$= \frac{4}{32} + \frac{2}{8} + \frac{12}{8} - \frac{16}{32} - \frac{6}{8} + \frac{4}{32}$$

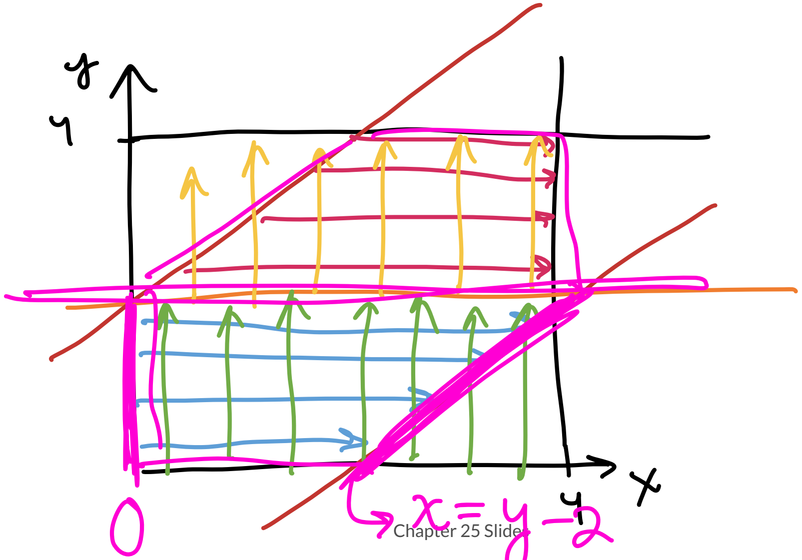
$$= \frac{1}{8} + \frac{2}{8} + \frac{12}{8} - \frac{4}{8} - \frac{6}{8} + \frac{1}{8} = \frac{6}{8} = \boxed{\frac{3}{4}}$$

ALT equal density across domain

$f_X(x)$ is same



ANOTHER ALT



$$P(|X-Y|<2) = \int_0^2 \int_0^{y+2} \frac{1}{16}\,dx\,dy + \int_2^4 \int_{y-2}^4 \frac{1}{16}\,dx\,dy$$

Finding the pdf of a transformation

- Let M be a transformation of X and Y
- When we have a transformation of X and Y, M, we need to follow a specific process to find the pdf of M

We follow this process:

1. Start with the joint pdf for X and Y ✓
    - aka $f_{X,Y}(x, y)$
2. Translate the domain of X and Y to M ✓
3. Find the CDF of M ✓
    - aka $F_M(m)$ or $P(M \leq m)$
4. Take the derivative of the CDF of M to find the pdf of M
    - aka $f_M(m) = \frac{d}{dm} F_M(m)$

Let’s complicate this even more!

Example 3.2

Let X and Y have constant density on the square $0 \le X \le 4, 0 \le Y \le 4$.

2. Let $M = \max(X, Y)$. Find the pdf for M, that is $f_M(m)$.

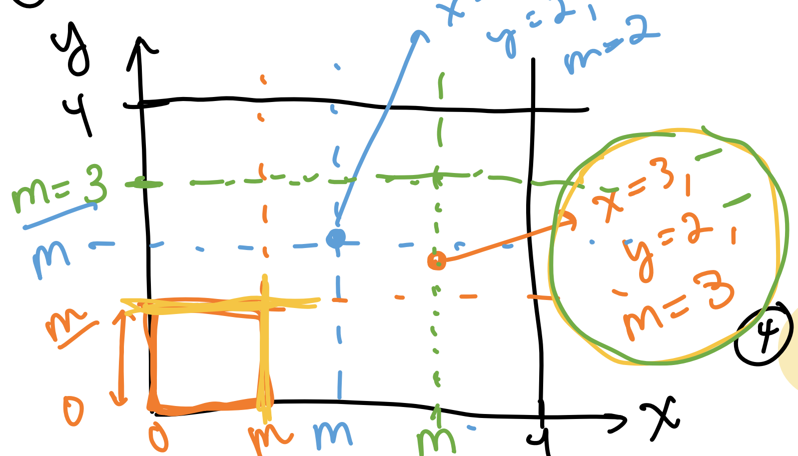
① $f_{X,Y}(x,y) = \frac{1}{16} \quad 0 \le x \le 4, \; 0 \le y \le 4$

$f_{x,y}(x,y) = k$

$\left[\int_0^4 \int_0^4 k \, dy \, dx = 1 \right]$

$\left[\underbrace{4}_{x} \times \underbrace{4}_{y} \times \underbrace{k}_{pdf} = 1 \right]$

② Domain



③ $F_M(m) = P(M \le m)$

$= P(\max(X, Y) \le m)$

$= P(X \le m, Y \le m)$

$= \int_0^m \int_0^m \frac{1}{16} \, dy \, dx = \int_0^m \left[\frac{y}{16} \right]_{y=0}^{m} dx$

$= \int_0^m \underbrace{\frac{m}{16}}_{\text{constant}} dx = \frac{m}{16} x \Big|_{x=0}^{x=m} = m^2/16$

④ $f_M(m) = \frac{d}{dm} F_M(m) = \frac{d}{dm} \frac{m^2}{16} = \frac{2m}{16} = \frac{1}{8} m$ for $0 \le m \le 4$

Let's complicate this even more!

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Example 3.3

Let X and Y have constant density on the square $0 \leq X \leq 4, 0 \leq Y \leq 4.$

3. Let $Z = \min(X, Y)$. Find the pdf for Z, that is $f_Z(z)$.

Let's complicate this even further!

Do this problem at home for extra practice. I'll add the solution to the annotated notes!

Example 4

Let X and Y have joint density $f_{X,Y}(x, y) = \frac{8}{5}(x + y)$ in the region $0 < x < 1,\ \frac{1}{2} < y < 1$. Find the pdf of the r.v. Z, where $Z = XY$.