Chapter 26: Independent Continuous RVs

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Learning Objectives

1. Show that a joint pdf consists of two independent, continuous RVs.
2. Combine two independent RVs into one joint pdf or CDF.

How do we represent independent continuous RVs in a joint pdf?

What do we know about independence for events and discrete RVs?

For events: If $A \perp B$

$$P(A \cap B) = P(A)P(B)$$

$$P(A|B) = P(A)$$

For discrete RVs: If $X \perp Y$

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

$$p_{X|Y}(x|y) = p_X(x)$$

$$p_{Y|X}(y|x) = p_Y(y)$$

What does it mean for continuous r.v.'s to be independent?

For continuous RVs: If $X \perp Y$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$F_{X,Y}(x,y) = F_X(x) F_Y(y)$$

$$f_{X|Y}(x|y) = f_X(x)$$

$$f_{Y|X}(y|x) = f_Y(y)$$

Constructing a joint pdf from two independent, continuous RVs

Example 1.1

Let X and Y be independent r.v.'s with $f_X(x) = \frac{1}{2}$, for $0 \leq x \leq 2$ and $f_Y(y) = 3y^2$, for $0 \leq y \leq 1$.

1. Find $f_{X,Y}(x, y)$.

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \quad [\text{b/c } X \perp Y]$$

$$= \frac{1}{2} \cdot 3y^2$$

$$f_{X,Y}(x,y) = \frac{3}{2} y^2 \text{ for } \begin{array}{l} 0 \leq x \leq 2 \\ 0 \leq y \leq 1 \end{array}$$

domain of x & y ind?

- ex domain is NOT ind:

  $0 \leq x \leq y/2$

- we know domain is ind b/c the prob said $X \perp Y$

Prob from

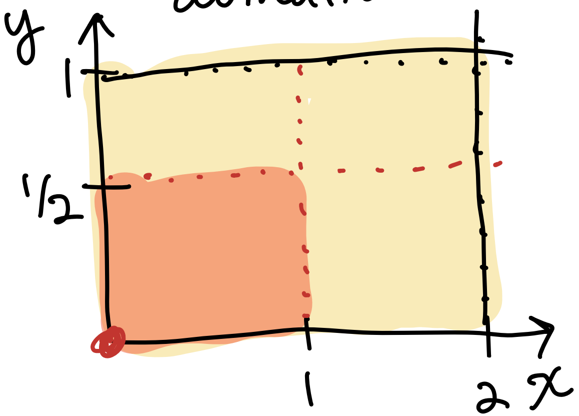
~~Constructing~~ a joint pdf from two independent, continuous RVs

Example 1.2

Let X and Y be independent r.v.'s with $f_X(x) = \frac{1}{2}$, for $0 \leq x \leq 2$ and $f_Y(y) = 3y^2$, for $0 \leq y \leq 1$.

2. Find $\mathbb{P}(0 \leq X \leq 1, 0 \leq Y \leq \frac{1}{2})$.

domain



$$P(0 \leq X \leq 1, 0 \leq Y \leq \tfrac{1}{2})$$

$$= \int_0^{1/2}\int_0^1 f_{X,Y}(x,y)\,dx\,dy \quad \text{(opt 1)}$$

best opt when NOT ind

$$= F(x=1, y=\tfrac{1}{2}) - F(x=0, y=0) \quad \text{(opt 2)}$$

$$\left[P(X \leq 1, Y \leq \tfrac{1}{2}) - \underbrace{P(X \leq 0, Y \leq 0)}_{=0}\right]$$

$$= F(x=1, y=\tfrac{1}{2})$$

$$= F_X(x=1)\,F_Y(y=\tfrac{1}{2})$$

$$= \left[\int_0^1 \underbrace{\tfrac{1}{2}}_{f_X(x)} dx\right]\left[\int_0^{1/2} \underbrace{3y^2}_{f_Y(y)} dy\right]$$

$$= \left[\tfrac{1}{2}x \Big|_0^1\right]\left[y^3\Big|_0^{1/2}\right] = \left[\tfrac{1}{2}(1) - \tfrac{1}{2}(0)\right] \cdot \left[\tfrac{1}{2}^3 - 0^3\right]$$

$$P(0 \leq X \leq 1, 0 \leq Y \leq 1/2) = \frac{1}{16}$$

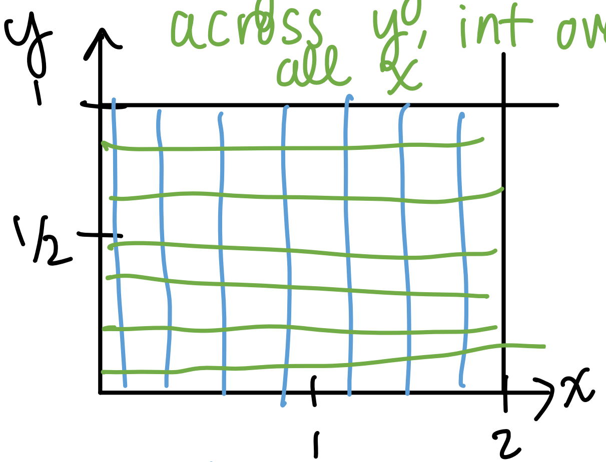
Showing independence from joint pdf

Example 2.1

Let $f_{X,Y}(x,y) = 18x^2y^5$, for $0 \le x \le 1,\ 0 \le y \le 1$.

1. Are X and Y independent?

marg of y means across y, int over all x



marg of x mean int over all y

- Need to show $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ and that domain for marginals are same as joint
- Need to find marginals!

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$f_X(x) = \int_{y=0}^{y=1} 18x^2y^5\,dy = 3x^2y^6 \Big|_{y=0}^{y=1}$$

$$= 3x^2(1)^6 - 3x^2(0)^6 \Rightarrow f_X(x) = 3x^2 \text{ for } 0 \le x \le 1$$

$$f_Y(y) = \int_{x=0}^{x=1} 18x^2y^5\,dx = 6x^3y^5 \Big|_{x=0}^{x=1}$$

$$= 6(1)^3y^5 - 6(0)^3y^5 \Rightarrow f_Y(y) = 6y^5 \text{ for } 0 \le y \le 1$$

$$f_X(x) f_Y(y) = 3x^2 \cdot 6y^5 = 18x^2y^5$$

✓ $f_{X,Y}(x,y) = 18x^2y^5 = f_X(x) f_Y(y)$ for $0 \le x \le 1$, $0 \le y \le 1$

Finding CDF from two independent RVs

Example 2.2

Let $f_{X,Y}(x,y) = 18x^2y^5$, for $0 \le x \le 1,\ 0 \le y \le 1$.

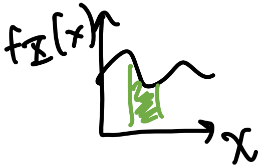
2. Find $F_{X,Y}(x,y)$.

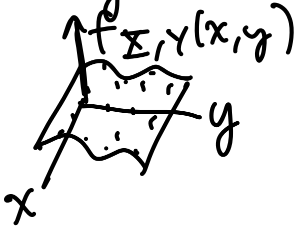
Steps

1. given joint pdf
2. get marg pdf's

done this!

3. int marg pdf's to get marg CDFs
4. multiply marg CDFs





③ $$F_X(x) = P(X \le x) = \int_0^x f_X(s)\,ds = \int_0^x 3s^2\,ds = s^3\Big|_{s=0}^{s=x} = x^3 \text{ for } 0 \le x \le 1$$

not comp CDFs

$$F_Y(y) = P(Y \le y) = \int_0^y f_Y(t)\,dt = \dots = y^6 \text{ for } 0 \le y \le 1$$

④ $$F_{X,Y}(x,y) = F_X(\overset{0}{x})\,F_Y(\overset{0}{y})$$

$$= \left[\begin{cases} 0 & x < 0 \\ x^3 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}\right]\left[\begin{cases} 0 & y < 0 \\ y^6 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}\right]$$

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ x^3y^6 & 0 \le x \le 1 \ \&\ 0 \le y \le 1 \\ x^3 & 0 \le x \le 1 \ \&\ y > 1 \\ y^6 & 0 \le y \le 1 \ \&\ x > 1 \\ 1 & x > 1 \ \&\ y > 1 \end{cases}.$$

Showing independence from joint pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 3

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$. Are X and Y independent?

Final statement on independence

1. If $f_{X,Y}(x,y) = g(x)h(y)$, where $g(x)$ and $h(y)$ are pdf's, then X and Y are independent.
   - The domain of the joint pdf needs to be independent as well!!

2. If $F_{X,Y}(x,y) = G(x)H(y)$, where $G(x)$ and $H(y)$ are cdf's, then X and Y are independent.
   - The domain of the joint CDF needs to be independent as well!!

x domain does NOT depend on y

y domain does NOT depend on x