Chapter 27: Conditional Distributions

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Learning Objectives

1. Calculate the conditional probability density from a joint pdf

Conditional probabilities we've seen before

What do we know about conditional probabilities for events and discrete RVs?

For events:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

For discrete RVs:

$$p_{X|Y}(x|y) = P(X = x | Y = y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

What does it mean for conditional densities of continuous RVs?

For continuous RVs:

$$\frac{\text{joint pdf}}{\text{marginal}}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Example starting from a joint pdf: first try!

Example 1.1

Let $f_{X,Y}(x,y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

1. Find $\mathbb{P}(2 < X < 10|Y = 4)$

WAY NOT TO DO THIS:

$$P(2 < X < 10 \mid Y=4) = \frac{P(2<X<10 \;\&\; Y=4)}{P(Y=4)}$$

for cont RV, Y,

$$P(Y=y) = 0$$

RIGHT WAY

we need to find $f_Y(y)$

then find $f_{X|Y}(x|y)$

then we can calc a probability

What is a conditional density?

Definition: Conditional density

The conditional density of a r.v. X given $Y = y$, is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)},$$

for $f_Y(y) > 0$

Remarks

1. It follows from the definition for the conditional density $f_{X|Y}(x|y)$, that

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y).$$

2. For a fixed value of $Y = y$, the conditional density $f_{X|Y}(x|y)$ is an actual pdf, meaning
   - $f_{X|Y}(x|y) \geq 0$ for all x and y, and
   - $\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = 1.$

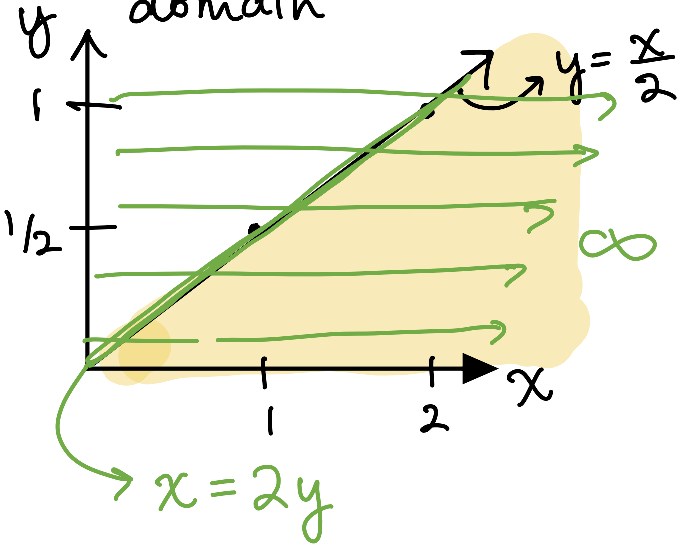
Example starting from a joint pdf: second try! (1/2)

Example 1.1

Let $f_{X,Y}(x,y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

1. Find $\mathbb{P}(2 < X < 10 | Y = 4)$

domain



$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{2y}^{\infty} 5e^{-x-3y}\,dx = \int_{2y}^{\infty} 5e^{-x}e^{-3y}\,dx$$

const.

$$= 5e^{-3y}\int_{2y}^{\infty} e^{-x}\,dx = \dots = 5e^{-5y},\ y>0$$

$$= \frac{5e^{-x}e^{-3y}}{5e^{-5y}} = e^{-x}e^{-3y-(-5y)} = e^{-x}e^{2y}$$

$$f_{X|Y}(x|y) = e^{-x}e^{2y} \quad 0 < y < \frac{x}{2}$$

$$\left[f_{X|Y}(x|y=4) = e^{-x}e^{8} \text{ for } x > 8 \right]$$

domain: $0 < 4 < \frac{x}{2}$, $4 < x/2$, $8 < x$

$$P(2 < X < 10 | Y=4) = \int_{2 \to 8}^{10} f_{X|Y}(x|y=4)\,dx$$

$$= \int_{8}^{10} e^{-x}e^{8}\,dx \Rightarrow P(2<X<10|Y=4) = 1 - e^{-2}$$

Example starting from a joint pdf: second try! (2/2)

why not $F_{X,Y}(x=10, y=4) - F_{X,Y}(\underline{\cancel{x=2}}, y=4)$
$x=8$

$$= \int_8^{10} \int_4^4 f_{X,Y}(x,y)\,dy\,dx$$

↳ ends up $= 0$

when $P(\_\_\_\_ \mid X=x)$, must find $f_{Y|X}(y \mid X=x)$ first!

Example starting from a joint pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 1.2

Let $f_{X,Y}(x, y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

2. Find $\mathbb{P}(X > 20|Y = 5)$

Finding probability with conditional domain and pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 2

Randomly choose a point X from the interval $[0, 1]$, and given $X = x$, randomly choose a point Y from $[0, x]$. Find $\mathbb{P}(0 < Y < \frac{1}{4})$.

Independence and conditional distributions

Question What is $f_{X|Y}(x|y)$ if X and Y are independent?

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_y(y)} = \frac{f_X(x)f_y(y)}{f_y(y)} = f_X(x)$$

- If $f_{X|Y}(x|y)$ does not depend on y (including the bounds/domain), then X and Y are independent.