Chapter 43: Moment Generating Functions

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Learning Objectives

1. Learn the definition of a moment-generating function.
2. Find the moment-generating function of a binomial random variable.
3. Use a moment-generating function to find the mean and variance of a random variable.

What are moments?

Definition 1

The j^{th} moment of a r.v. X is $\mathbb{E}[X^j]$

Example 1

$1^{st} - 4^{th}$ moments

What is a *moment generating function* (mgf)??

Definition 3

If X is a r.v., then the **moment generating function** (**mgf**) associated with X is:

$$M_X(t) = \mathbb{E}[e^{tX}]$$

Remarks

- For a discrete r.v., the mgf of X is

$$M_X(t) = \mathbb{E}[e^{tX}] = \sum_{all\ x} e^{tx} p_X(x)$$

- For a continuous r.v., the mgf of X is

$$M_X(t) = \mathbb{E}[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx$$

- The mgf $M_X(t)$ is a function of t, not of X, and it might not be defined (i.e. finite) for all values of t. We just need it to be defined for $t = 0$.

Example

Example 4

What is $M_X(t)$ for $t = 0$?

Theorem

Theorem 5

The moment generating function uniquely specifies a probability distribution.

Theorem 6

$$\mathbb{E}[X^r] = M_X^{(r)}(0)$$

(r) in this equation is the rth derivative with respect to t

- When $r = 1$, we are taking the first derivative
- When $r = 4$, we are taking the fourth derivative

Using the mgf to uniquely describe a probability distribution

Example 7

Let $X \sim Poisson(\lambda)$

1. Find the mgf of X
2. Find $\mathbb{E}[X]$
3. Find $Var(X)$

Theorem

Remark: Finding the mean and variance is sometimes easier with the following trick

Theorem 8

Let $R_X(t) = \ln[M_X(t)]$. Then,

$$\mu = \mathbb{E}[X] = R'_X(0), \text{ and}$$

$$\sigma^2 = Var(X) = R''_X(0)$$

Proof.

Using $R_X(t)$ to uniquely describe a probability distribution

Example 9

Let $X \sim Poisson(\lambda)$.

1. Find $\mathbb{E}[X]$ using $R_X(t)$
2. Find $Var(X)$ using $R_X(t)$

Using the mgf to uniquely describe the standard normal distribution

Example 10

Let Z be a standard normal random variable, i.e. $Z \sim N(0, 1)$.

1. Find the mgf of Z
2. Find $\mathbb{E}[Z]$
3. Find $Var(Z)$

Mgf's of sums of independent RV's

Theorem 9

If X and Y are independent RV's with respective mgf's $M_X(t)$ and $M_Y(t)$, then

$$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX}e^{tY}] = E[e^{tX}]E[e^{tY}] = M_X(t)M_Y(t)$$

Main takeaways

- Mgf's are a purely mathematically definition
  - We can't really relate it to our real world analysis
- They are helpful mathematically because they are unique to a probability distribution
  - We can find the unique mgf from for a probability distribution
  - And we can find a distribution from an mgf
- Mgf's can *sometimes* make it easier to find the mean and variance of an RV
- Mgf's are most helpful when we are finding a joint distribution that is a sum or transformation of two RV's
  - Make the calculation easier!
- Mgf's are often used to prove certain distribution are sums of other ones!

More resources

- https://online.stat.psu.edu/stat414/book/export/html/676
- https://www.youtube.com/watch/ez_vq23xWrQ
- https://www.youtube.com/watch/2p9J9ChTeFI
- https://www.youtube.com/watch/A5bWU8xcQkE
- https://www.youtube.com/watch/QeUrTGFTFm4
- https://www.youtube.com/watch/HhrkwyyRtgI