Lesson 7: Poisson distribution

TB sections 3.4

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Learning Objectives

1. Calculate probabilities for different events using a Poisson distribution

Where are we?

Data	Probability	Sampling Variability, and Statistical Inference
Collecting data	Probability rules	Sampling distributions
Categorical vs. Numeric	Independence, conditional	Central Limit Theorem
Summary statistics	Random variables and probability distributions	Confidence Intervals
Data visualization	Linear combinations	Hypothesis tests
	Binomial, Normal, and Poisson	

Inference for continuous data/outcomes

One sample t-test	3+ independent samples	Simple linear regression / correlation
2 sample tests: paired and independent	Power and sample size	Non-parametric tests

Inference for categorical data/outcomes

One proportion test	Fisher's exact test	Non-parametric tests
Chi-squared test	2 proportion test	Power and sample size

R							
Basics	Reproducibility	Quarto	Packages	Data visualization	Data wrangling	R Projects	…

Introduction to the Poisson distribution

- The Poisson distribution is often used to model **count data (# of successes)**, especially for **rare events**
  - It is a **discrete distribution!**

- It is used most often in settings where **events happen at a rate λ** per unit of population and per unit time

- **Example:** historical records of hospitalizations in New York City indicate that an average of 4.4 people are hospitalized each day for an acute myocardial infarction (AMI)
  - We can plot the distribution of hospitalizations on each day

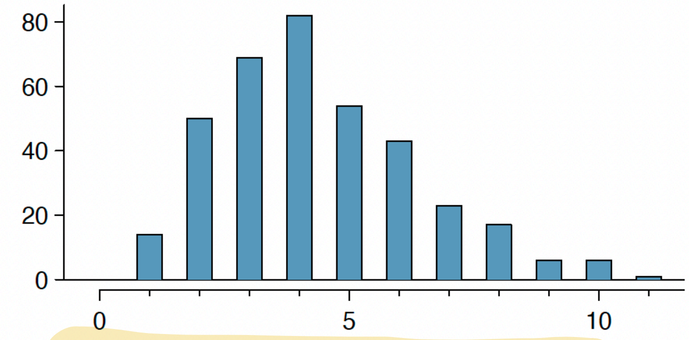


Figure 3.20: A histogram of the number of people hospitalized for an AMI on 365 days for NYC, as simulated from a Poisson distribution with mean 4.4.

Poisson distribution

prob of hosp in 1 min
is proportional to
prob of hosp in 1 hour

- Suppose events occur over time in such a way that
  1. The probability an event occurs in an interval is proportional to the length of the interval.
  2. Events occur independently at a rate λ per unit of time.

$$\lambda = \frac{0.2 \text{ hosp}}{1 \text{ minute}} = \frac{12 \text{ hosp}}{1 \text{ hour}}$$

$$\frac{0.2 \text{ hosp}}{1 \text{ min}} \times \frac{60 \text{ min}}{\text{hour}} = \frac{12 \text{ hosp}}{\text{hour}}$$

- Then the probability of exactly k events in one unit of time is

min → hours

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$4! = 4 \cdot 3 \cdot 2 \cdot 1$

- For the Poisson distribution modeling the number of events in one unit of time:
  - The mean is λ.
  - The standard deviation is $\sqrt{\lambda}$.

variance is λ

- Shorthand for a random variable, X, that has a Poisson distribution:

$$X \sim \text{Pois}(\lambda)$$

"distributed as"

Poisson distribution: R commands

R commands with their input and output:

R code	What does it return?	
`rpois()`	returns sample of random variables with specified Poisson distribution	→ sample count
`dpois()`	returns value of probability density at certain point of the Poisson distribution	$P(X = k)$
`ppois()`	returns cumulative probability of getting certain point (or less) of the Poisson distribution	$P(X \le k)$
`qpois()`	returns number of cases corresponding | to desired quantile	$P(X \le q) = 0.9$

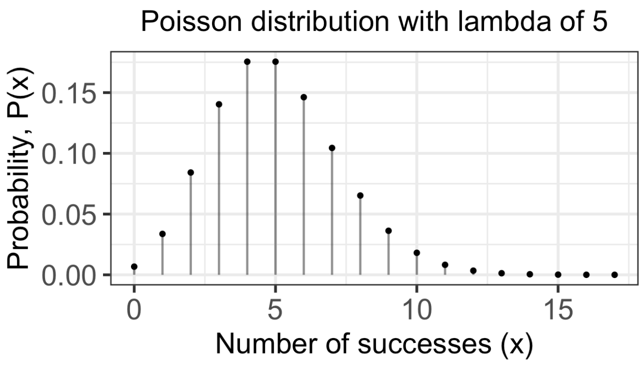
Example: probabilities from a Poisson distribution (1/4)

Typhoid fever

mean of Poisson is λ, $\lambda = 5/yr$

Suppose there are on average 5 deaths per year from typhoid fever over a 1-year period.

1. What is the probability of 3 deaths in a year?
2. What is the probability of 2 deaths in 0.5 years?
3. What is the probability of more than 12 deaths in 2 years?



let X = # deaths per yer

$\lambda = 5/yr$

Example: probabilities from a Poisson distribution (2/4)

Typhoid fever

Suppose there are on average 5 deaths per year from typhoid fever over a 1-year period.

1. What is the probability of 3 deaths in a year? $\lambda = 5$

- $\lambda = 5$ and we want $P(X = 3)$

$$P(X = 3) = \frac{e^{-5} 5^3}{3!} = 0.1404$$

```
dpois(x = 3, lambda = 5)
[1] 0.1403739
```

$= P(X=3)$

The prob of 3 deaths in a year from typhoid fever is 0.14

Example: probabilities from a Poisson distribution (3/4)

Typhoid fever

Suppose there are on average 5 deaths per year from typhoid fever over a 1-year period.

2. What is the probability of 2 deaths in 0.5 years?

$\lambda = ?$ and we want $P(X=2)$

let X = # deaths in 0.5 yrs

$\lambda = \frac{\_\_ \text{ deaths}}{0.5 \text{ yrs}}$

- $\lambda = 5$ was the rate for one year. When we want the rate for half year, we need to calculate a new λ:
  - $\lambda = \frac{5 \text{ deaths}}{1 \text{ year}} \cdot \frac{1 \text{ year}}{2 \text{ half-years}} = \frac{2.5 \text{ deaths}}{1 \text{ half-year}}$

$$P(X=2) = \frac{e^{-2.5} 2.5^2}{2!} = 0.2565$$

```
dpois(x = 2, lambda = 2.5)
```

```
[1] 0.2565156
```

always need to change λ

not THE SAME!!

Is this the same as $P(X=4)$ w/ $\lambda = 5$

vs $P(X=2)$ w/ $\lambda = 2.5$

Example: probabilities from a Poisson distribution (4/4)

Typhoid fever

Suppose there are on average 5 deaths per year from typhoid fever over a 1-year period.

3. What is the probability of more than 12 deaths in 2 years?

means > 12

$\lambda = ?$ and we want $P(X > 12)$

- Need to calculate a new λ for 2 years: $\lambda = \dfrac{5 \text{ deaths}}{1 \cancel{\text{ year}}} \cdot \dfrac{2 \cancel{\text{ years}}}{1 \text{ two-years}} = \dfrac{10 \text{ deaths}}{1 \text{ two-year}} = \dfrac{10 \text{ deaths}}{2 \text{ years}}$

$$P(X > 12) = 1 - P(X \leq 12) = 1 - \sum_{k=0}^{12} \frac{e^{-10} 10^k}{k!} = 0.2084$$

$P(X=0) + P(X=1) + P(X=2) + \ldots + P(X=12)$

$1 - P(X \leq 12)$

```r
1 - ppois(q = 12, lambda = 10)
```

[1] 0.2084435

```r
ppois(q = 12, lambda = 10,
      lower.tail = F)
```

$P(X > 12)$

[1] 0.2084435

good for RARE events

Normal approx of binomial

Poisson approximation of binomial distribution

- Poisson distribution can be used to approximate binomial distribution when n is large and p is small
  - When Normal approximation does not work → large n & p close to 0.5
- Binomial distributions for different n (columns) and p (rows)