Lesson 11: Confidence intervals

TB sections 4.2

Nicky Wakim

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Learning Objectives

1. Calculate a confidence interval when we know the population standard deviation
2. Interpret a confidence interval when we know the population standard deviation
3. Calculate and interpret a confidence interval *using the t-distribution* when we do not know the population standard deviation

not usually available

typical!

Where are we?

Data
- Collecting data
- Categorical vs. Numeric
- Summary statistics
- Data visualization

Probability
- Probability rules
- Independence, conditional
- Random variables and probability distributions
- Linear combinations
- Binomial, Normal, and Poisson

Sampling Variability, and Statistical Inference
- Sampling distributions
- Central Limit Theorem
- Confidence Intervals
- Hypothesis tests

Inference for continuous data/outcomes
- One sample t-test
- 3+ independent samples
- Simple linear regression / correlation
- 2 sample tests: paired and independent
- Power and sample size
- Non-parametric tests

Inference for categorical data/outcomes
- One proportion test
- Fisher's exact test
- Non-parametric tests
- Chi-squared test
- 2 proportion test
- Power and sample size

R
- Basics
- Reproducibility
- Quarto
- Packages
- Data visualization
- Data wrangling
- R Projects
- …

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1. Calculate a confidence interval when we know the population standard deviation
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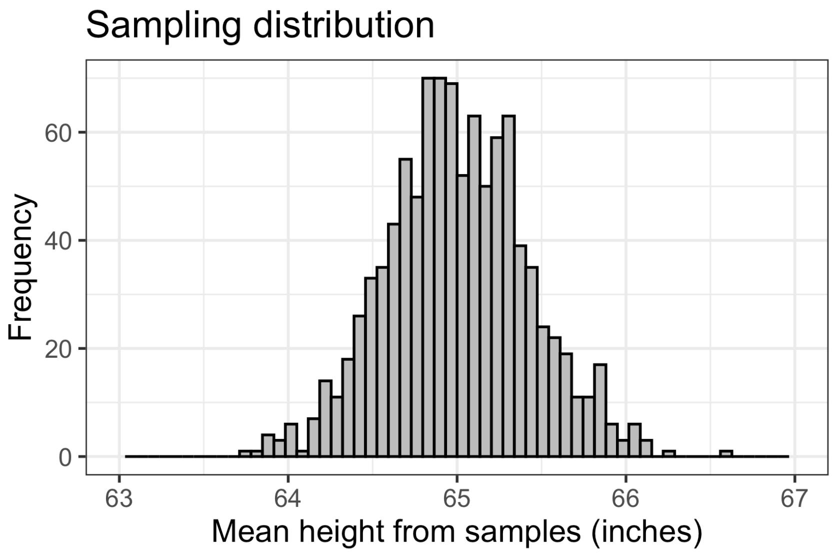
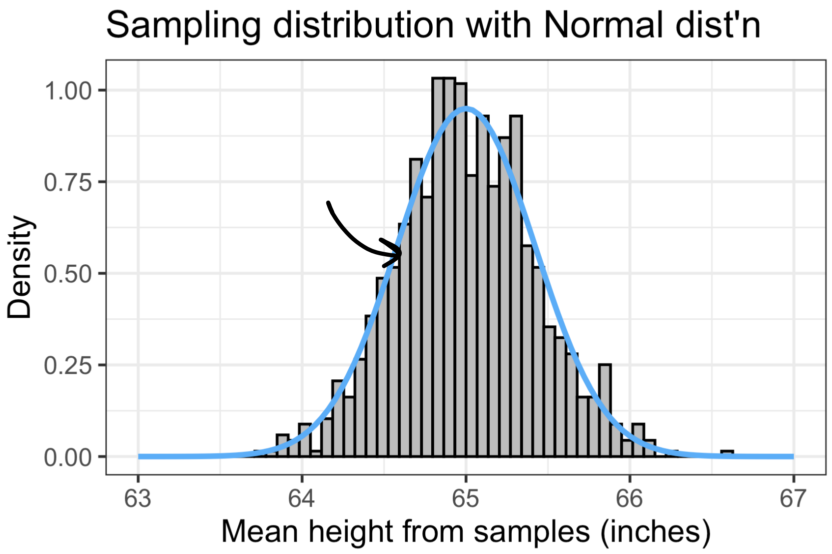
Last time: Central Limit Theorem applied to sampling distribution

- CLT tells us that we can model the sampling distribution of mean heights using a normal distribution





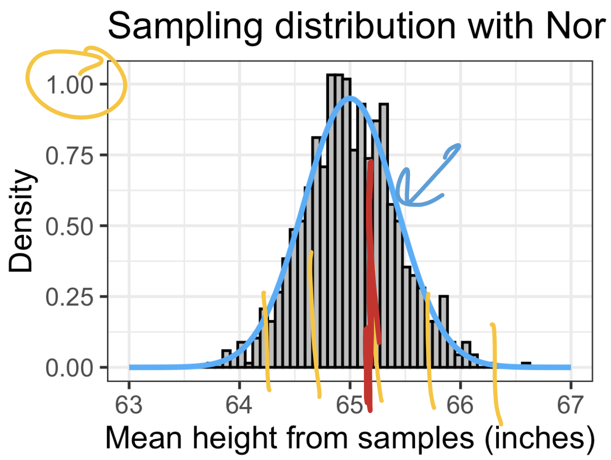
sampling dist approx Normal

$$\overline{X} \sim \text{Normal}\left(\mu_{\overline{X}} = 65, SE = 0.424\right)$$

w/ mean 65 & std error 0.424

Last time: Sampling Distribution of Sample Means (with the CLT)

- The **sampling distribution** is the distribution of sample means calculated from repeated random samples of *the same size* from the same population
- It is useful to think of a **particular sample statistic** as being **drawn from a sampling distribution**
  - So the red sample with $\overline{x} = 65.1$ is **just one sample mean** in the **sampling distribution**



$$\mu_{\overline{X}} = 65 \text{ inches}$$

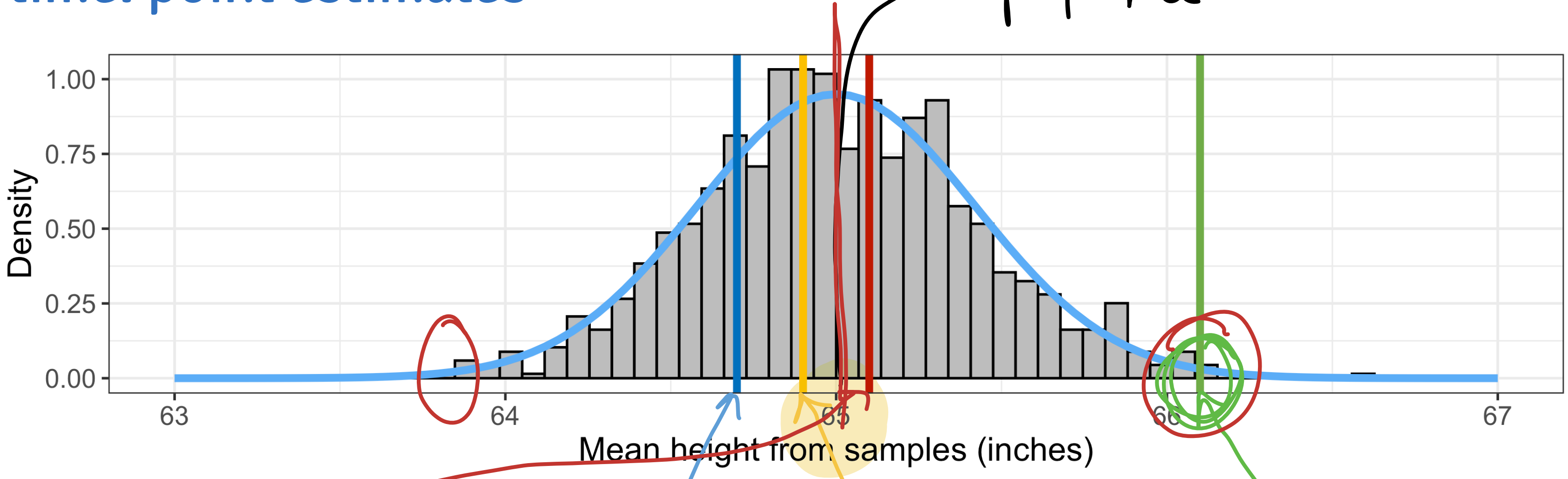
$$SE = 0.424 \text{ inches}$$

With CLT and $\overline{X}$ as the RV for the sampling distribution

- **Theoretically** (using only population values):
  $\overline{X} \sim \text{Normal}\left(\mu_{\overline{X}} = \mu, \sigma_{\overline{X}} = SE = \frac{\sigma}{\sqrt{n}}\right)$
- **In real use** (using sample values for SE):
  $\overline{X} \sim \text{Normal}\left(\mu_{\overline{X}} = \mu, \sigma_{\overline{X}} = SE = \frac{s}{\sqrt{n}}\right)$

Last time: point estimates

pop mean is 65



Sample 50 people
$\bar{x} = 65.1, s = 2.8$

Sample 50 people
$\bar{x} = 64.7, s = 3.1$

Sample 50 people
$\bar{x} = 64.9, s = 3.2$

Sample 50 people
$\bar{x} = 66.1, s = 3.4$

This time: Interval estimates of population parameter

- A **point estimate** consists of a single value
- An **interval estimate** provides a **plausible range of values for a parameter**
  - Remember: parameters are from the population and estimates are from our sample
- We can create a plausible range of values for a population mean (μ) from a sample's mean $\overline{x}$
- A **confidence interval** gives us a plausible range for μ
- Confidence intervals take the general form:

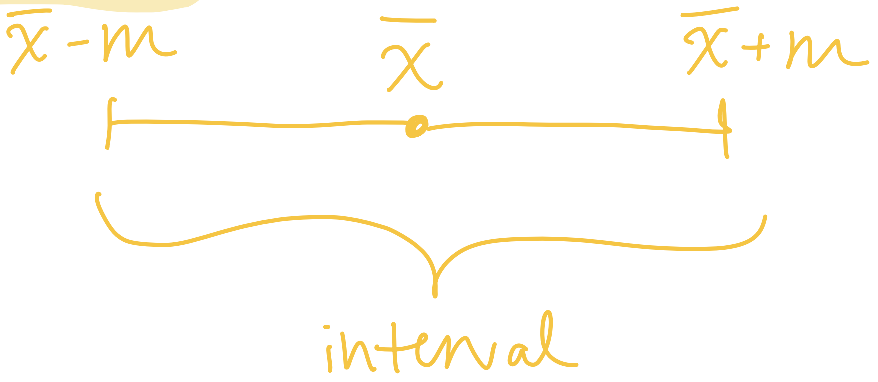
↳ 65

↳ red sample

$\bar{x} = 65.1$

$$(\overline{x} - m, \overline{x} + m) = \overline{x} \pm m$$

  - Where m is the margin of error



Point estimates with their confidence intervals for μ

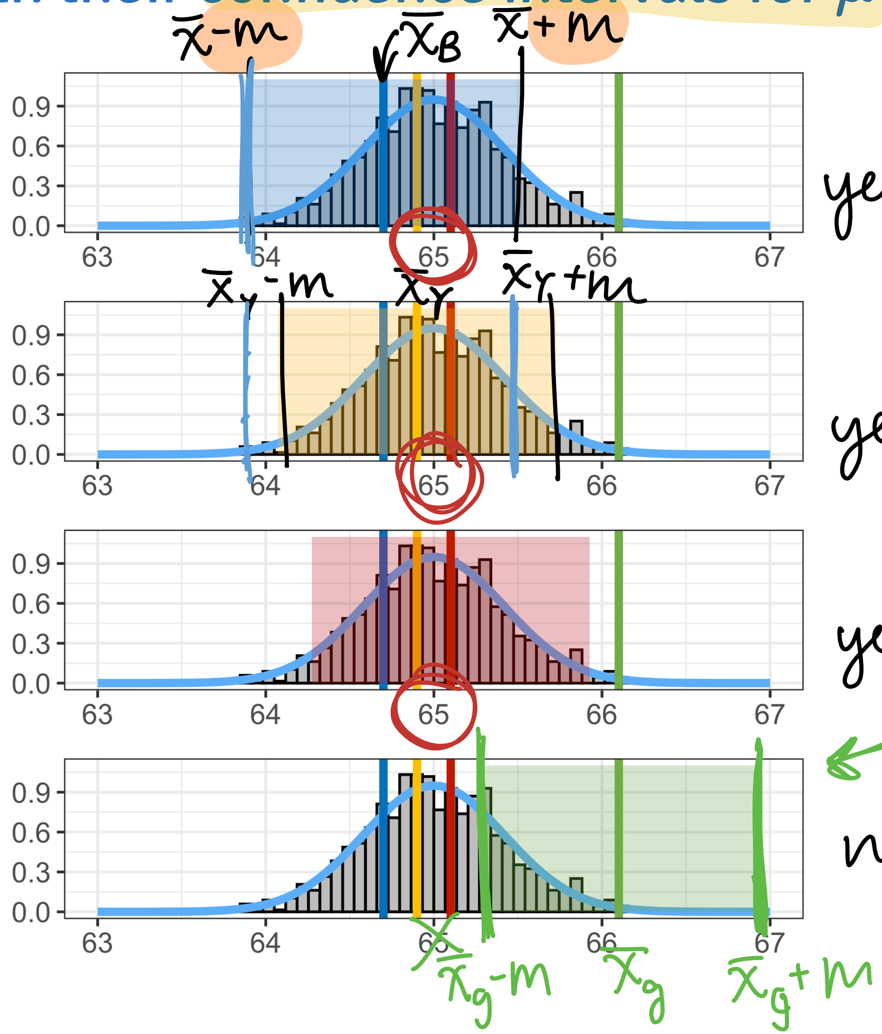
Do these confidence intervals include μ?

① Sample 50 people
$\bar{x} = 64.7, s = 3.1$

② Sample 50 people
$\bar{x} = 64.9, s = 3.2$

③ Sample 50 people
$\bar{x} = 65.1, s = 2.8$

④ Sample 50 people
$\bar{x} = 66.1, s = 3.4$



① yes

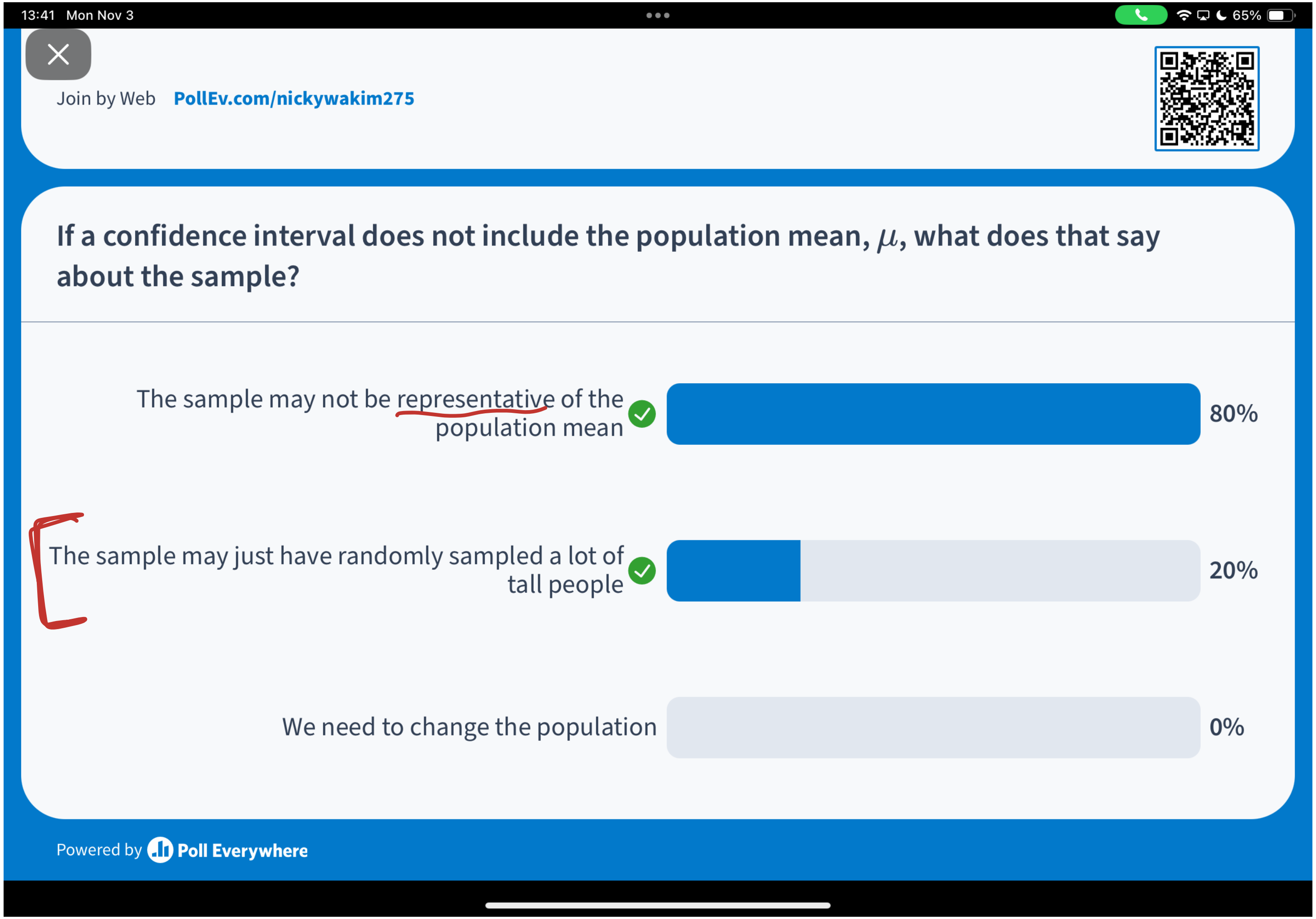
② yes

③ yes

④ no

65.268 to 66.932

Poll Everywhere Question 1



Confidence interval (CI) for the mean μ

Confidence interval for μ

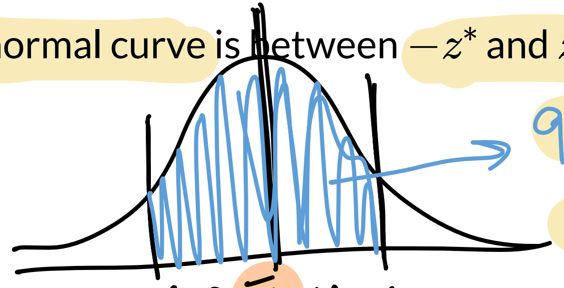
$$\overline{x} \pm z^* \times \text{SE}$$

- with $\text{SE} = \frac{\sigma}{\sqrt{n}}$ if population sd is known

When can this be applied?

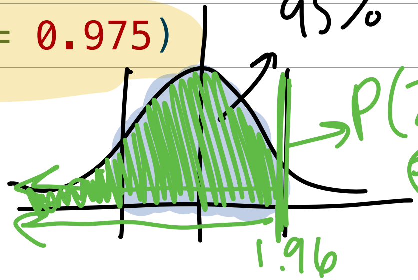
- When CLT can be applied!
- When we know the population standard deviation!

- z^* depends on the confidence level
- For a 95% CI, z^* is chosen such that 95% of the standard normal curve is between $-z^*$ and z^*
  - This corresponds to $z^* = 1.96$ for a 95% CI
- We can use R to calculate z^* for any desired CI
- Below is how we calculate z^* for the 95% CI




```
qnorm(p = 0.975)
```

```
[1] 1.959964
```



Example: CI for mean height μ with σ

$\bar{x} = 66.1$

Example 1: Using our green sample from previous plots

For a random sample of 50 people, the mean height is 66.1 inches. Assume the population standard deviation is 3 inches. Find the 95% confidence interval for the population mean.

$\sigma = 3$ $\quad$ $n = 50$

CI:

$$\bar{x} \pm z^* \times \text{SE}$$

$$\bar{x} \pm z^* \times \frac{\sigma}{\sqrt{n}}$$

$$66.1 \pm 1.96 \times \frac{3}{\sqrt{50}}$$

z^* for 95% CI

$$66.1 \pm 0.8315576$$

$$(66.1 - 0.8315576, 66.1 + 0.8315576)$$

$$(65.268, 66.932)$$

We are 95% confident that the population mean height is between 65.268 and 66.932 inches.

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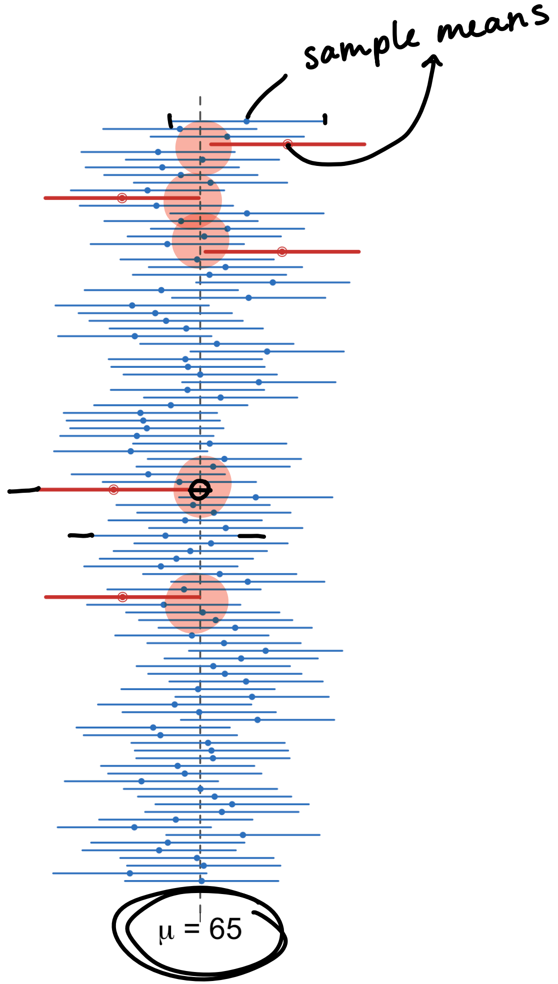
How do we interpret confidence intervals? (1/2)

Simulating Confidence Intervals:
http://www.rossmanchance.com/applets/ConfSim.html

The figure shows CI's from 100 samples:

- 100 samples: I find the mean and confidence interval of each sample
- The true value of $\mu = 65$ is the vertical black line
- The horizontal lines are 95% CI's from 100 samples
  - **Blue**: the CI "captured" the true value of μ
  - **Red**: the CI *did not* "capture" the true value of μ



What percent of CI's captured the true value of μ?

95/100 sample CI's captured μ

5/100 did not!

How do we interpret confidence intervals? (2/2)

Actual interpretation:

- If we were to
  - **repeatedly take random samples** from a population and
  - calculate a 95% CI for each random sample,
- then we would **expect 95% of our CI's to contain the true population parameter** μ.

What we typically write as "shorthand":

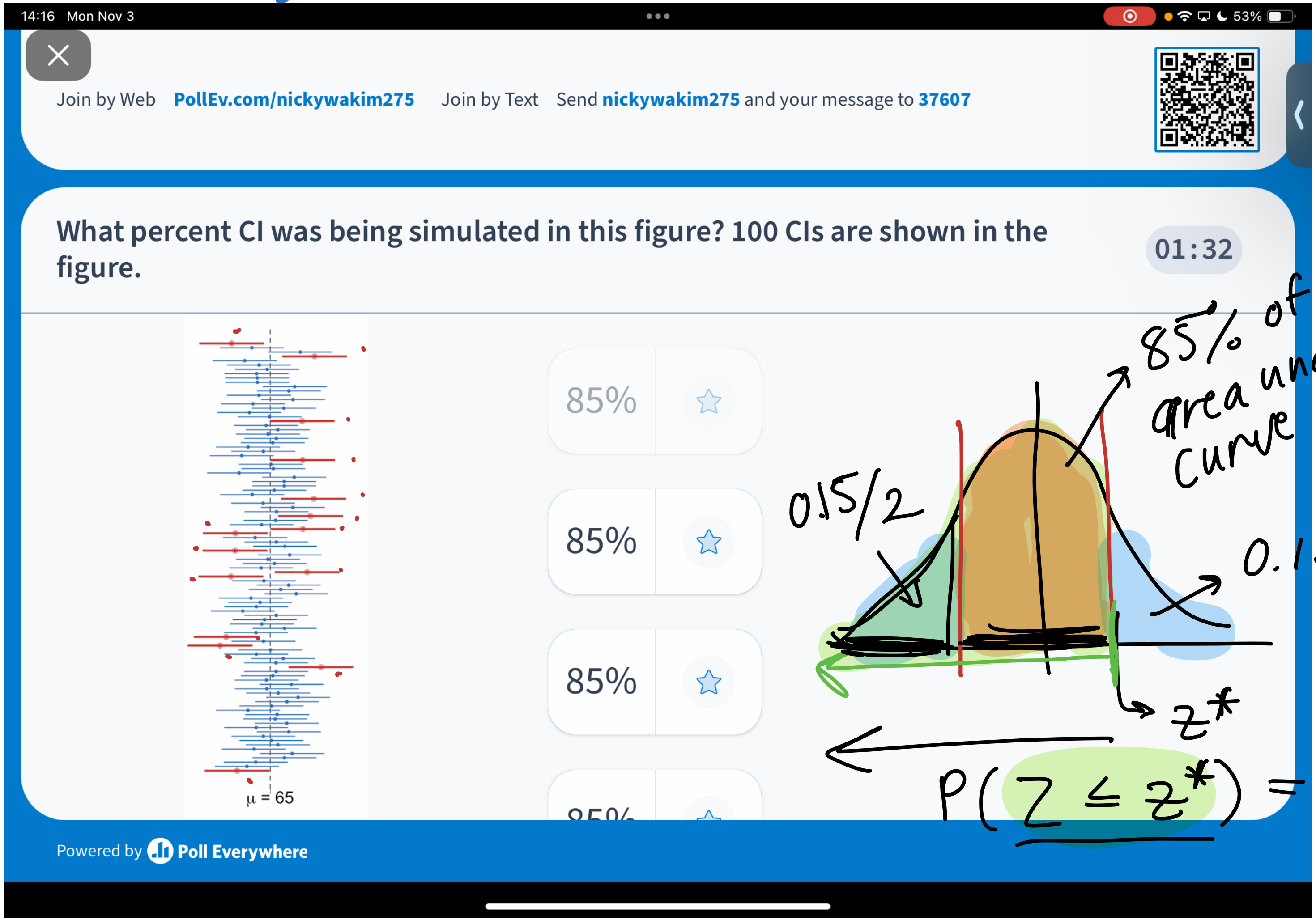
insert CI here

- In general form: We are 95% *confident* that (the 95% confidence interval) captures the value of the population parameter.
- Recall example: "We are 95% confident that the population mean height is between 65.268 and 66.932 inches."

WRONG interpretation:

- There is a 95% *chance* that (the 95% confidence interval) captures the value of the population parameter.
  - For one CI on its own, it either does or doesn't contain the population parameter with probability 0 or 1. We just don't know which!

Poll Everywhere Question 2



15 CIs do not capture μ

85 CIs do

85% CI

$z^* = ?$

$P(Z \le z^*) = 0.85 + \frac{0.15}{2}$

$z^* = 1.44$

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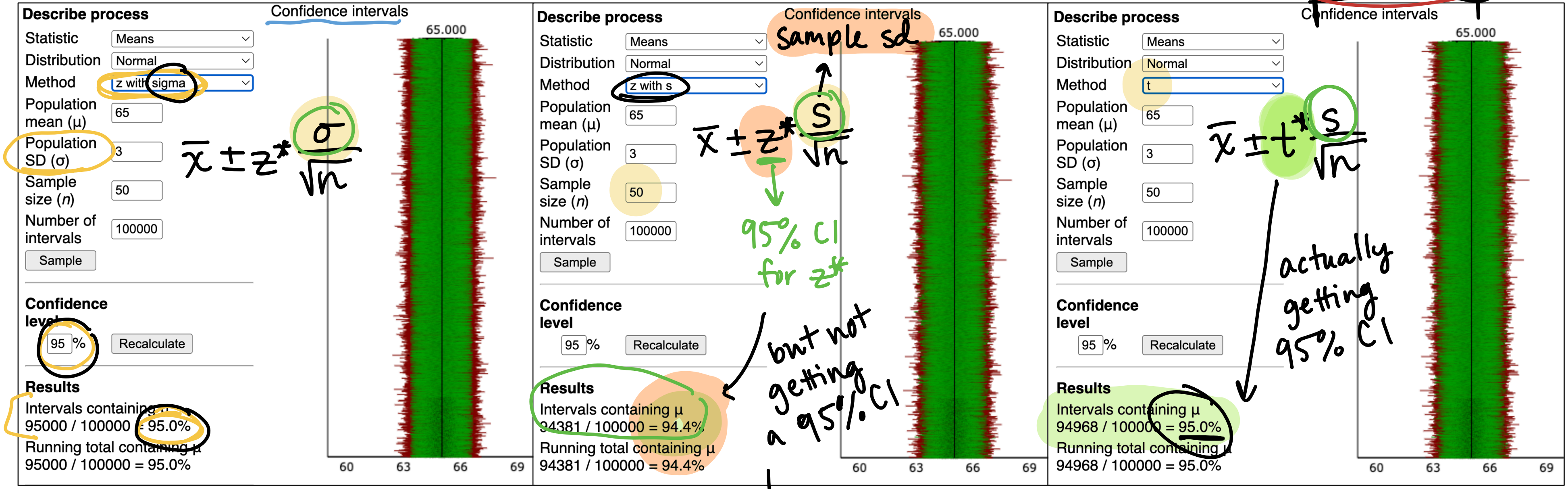
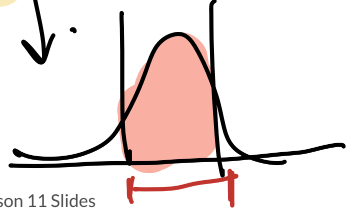
What if we don't know σ ? (1/2)

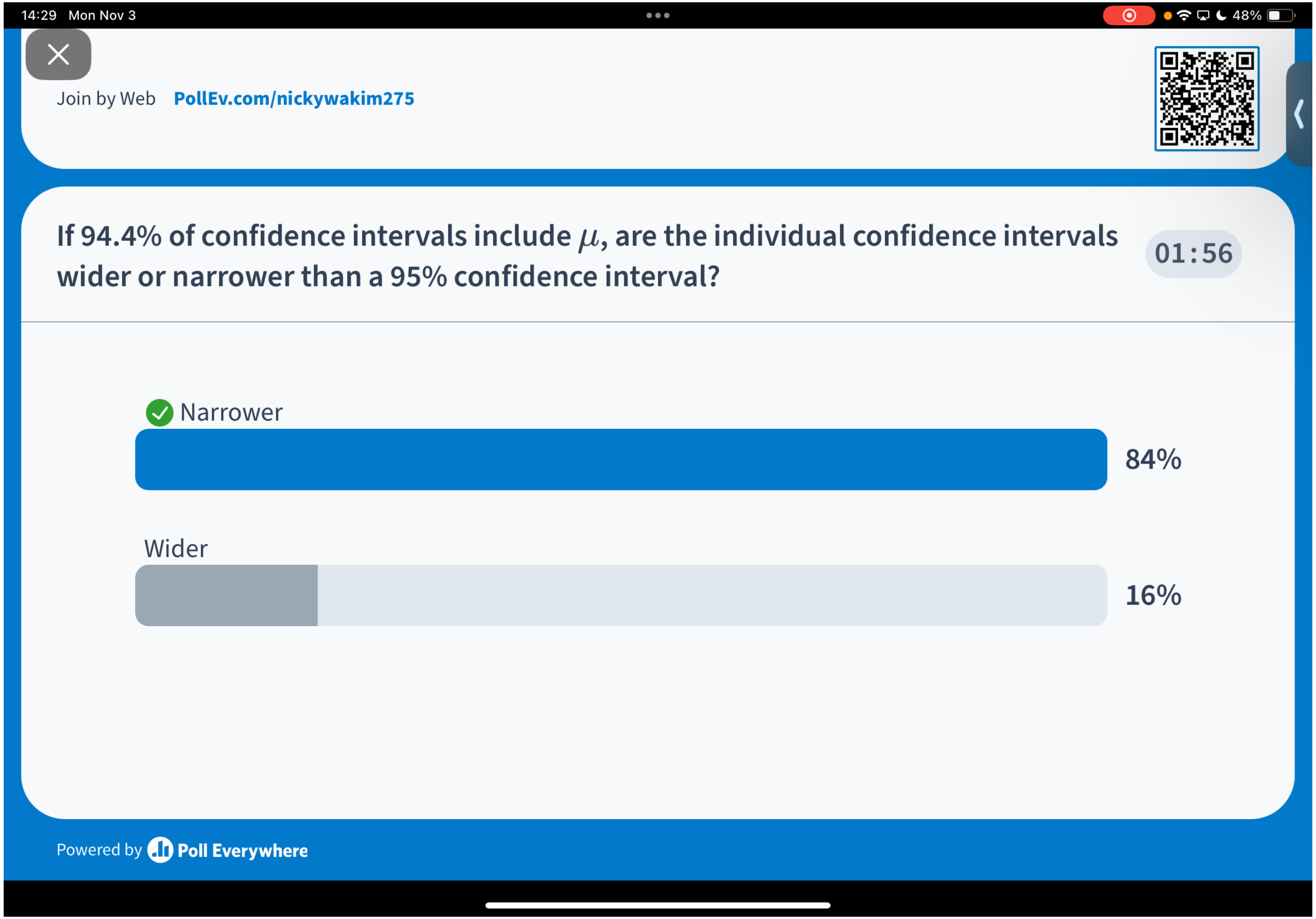
Simulating Confidence Intervals: http://www.rossmanchance.com/applets/ConfSim.html

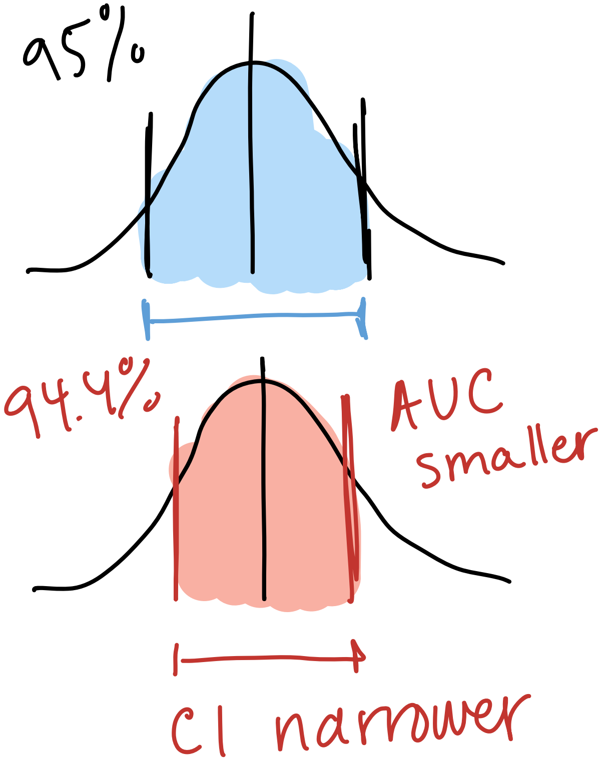


- The normal distribution doesn't have a 95% "coverage rate" when using s instead of σ
- There's another distribution, called the t-distribution, that does have a 95% "coverage rate" when we use s



Poll Everywhere Question 3





What if we don't know σ ? (2/2)

- In real life, we don't know what the population sd is (σ)
- If we replace σ with s in the SE formula, we add in additional variability to the SE!

$$\frac{\sigma}{\sqrt{n}} \quad \text{vs.} \quad \frac{s}{\sqrt{n}}$$

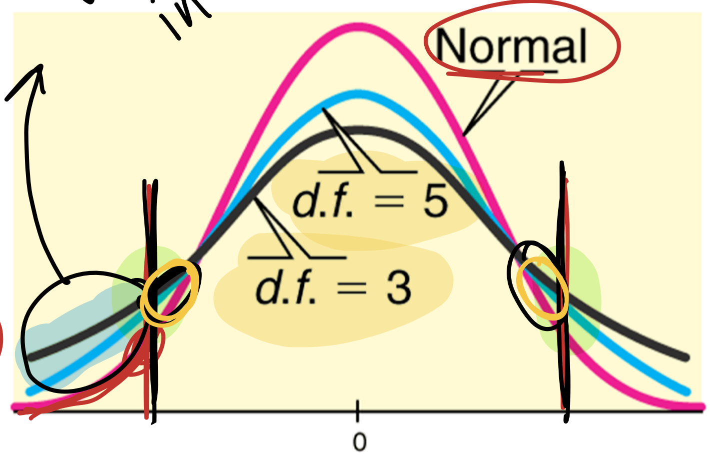
- Thus when using s instead of σ when calculating the SE, we **need a different probability distribution** with thicker tails than the normal distribution.
  - In practice this will mean using a different value than 1.96 when calculating the CI
- Instead, we use the **Student's t-distribution**

Student’s t-distribution

- Is bell shaped and symmetric
- A “generalized” version of the normal distribution
- Its tails are a thicker than that of a normal distribution
  - The “thickness” depends on its **degrees of freedom**: $df = n{-}1$, where n = sample size

when using s (sample sd)

more area under curve in tails (blue/black)



- As the degrees of freedom (sample size) increase,
  - the tails are less thick, and
  - the t-distribution is more like a normal distribution
  - in theory, with an infinite sample size the *t*-distribution is a normal distribution.

→ sample size → ∞

→ t → N

Confidence interval (CI) for the mean μ

Confidence interval for μ

$$\overline{x} \pm t^* \times \text{SE}$$

- with $\text{SE} = \frac{s}{\sqrt{n}}$ if population sd is not known

When can this be applied?

- When CLT can be applied!
- When we **do not** know the population standard deviation!

rt()
pt()
qt()
dt()

- t^* depends on the confidence level and degrees of freedom
  - degrees of freedom (df) is: $df = n - 1$ (n is number of observations in sample)
- `qt` gives the quartiles for a t-distribution. Need to specify
  - the percent under the curve to the left of the quartile
  - the degrees of freedom = $n - 1$
- Note in the R output to the right that t^* gets closer to 1.96 as the sample size increases

```
qt(p = 0.975, df=9)  #df = n-1
[1] 2.262157
```

$= t^*$

```
qt(p = 0.975, df=49)
[1] 2.009575
```

n=50

```
qt(p = 0.975, df=99)
[1] 1.984217
```

n=100

```
qt(p = 0.975, df=999)
[1] 1.962341
```

n=1000

↳ getting closer to z^*

Example: CI for mean height μ with s

$n = 50$ $\bar{x} = 66.1$ $s = 3.5$

Example 2: Using our green sample from previous plots

For a random sample of 50 people, the mean height is 66.1 inches and the standard deviation is 3.5 inches. Find the 95% confidence interval for the population mean.

$$\bar{x} \pm t^* \times \text{SE}$$

$$\bar{x} \pm t^* \times \frac{s}{\sqrt{n}}$$

$$66.1 \pm 2.0096 \times \frac{3.5}{\sqrt{50}}$$

$$66.1 \pm 0.994689$$

$$(66.1 - 0.994689, 66.1 + 0.994689)$$

$$(65.105, 67.095)$$

What is t^*?

$$df = n - 1 = 50 - 1 = 49$$

$t^* =$ `qt(p = 0.975, df = 49)` $= 2.0096$

We are 95% confident that the population mean height is between 65.105 and 67.095 inches.

Confidence interval (CI) for the mean μ (z vs. t)

- In summary, we have two cases that lead to different ways to calculate the confidence interval

Case 1: We know the population standard deviation

$$\overline{x} \pm z^* \times \text{SE}$$

- with $\text{SE} = \frac{\sigma}{\sqrt{n}}$ and σ is the population standard deviation
- For 95% CI, we use:
  - $z^* =$ `qnorm(p = 0.975)` $= 1.96$

Case 2: We do not know the population sd

$$\overline{x} \pm t^* \times \text{SE}$$

- with $\text{SE} = \frac{s}{\sqrt{n}}$ and s is the sample standard deviation
- For 95% CI, we use:
  - $t^* =$ `qt(p = 0.975, df = n−1)`

Some final words (said slightly differently?)

- Rule of thumb:
  - Use normal distribution ONLY if you know the population standard deviation σ
  - If using s for the SE, then use the Student's t-distribution

- For either case, we need to remember when we can calculate the confidence interval:
  - $n \geq 30$ and population distribution not strongly skewed (using Central Limit Theorem)
    - If there is skew or some large outliers, then $n \geq 50$ gives better estimates
  - $n < 30$ and data approximately symmetric with no large outliers

    ↳ normally distributed

- *If do not know population distribution, then check the distribution of the data.*
  - Aka, use what we learned in datavisualization to see what the data look like