Lesson 14: Power and sample size calculations for means

TB sections 5.4

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2024-11-20

Learning Objectives

1. Understand the four components in equilibrium in a hypothesis test.
2. Define the significance level, critical value, and rejection region.
3. Define power and understand its role in a hypothesis test.
4. Understand how to calculate power for two independent samples.
5. Using R, calculate power and sample size for a single mean t-test and two independent mean t-test.

Where are we?

Data
- Collecting data
- Categorical vs. Numeric
- Summary statistics
- Data visualization

Probability
- Probability rules
- Independence, conditional
- Random variables and probability distributions
- Linear combinations
- Binomial, Normal, and Poisson

Sampling Variability, and Statistical Inference
- Sampling distributions
- Central Limit Theorem
- Confidence Intervals
- Hypothesis tests

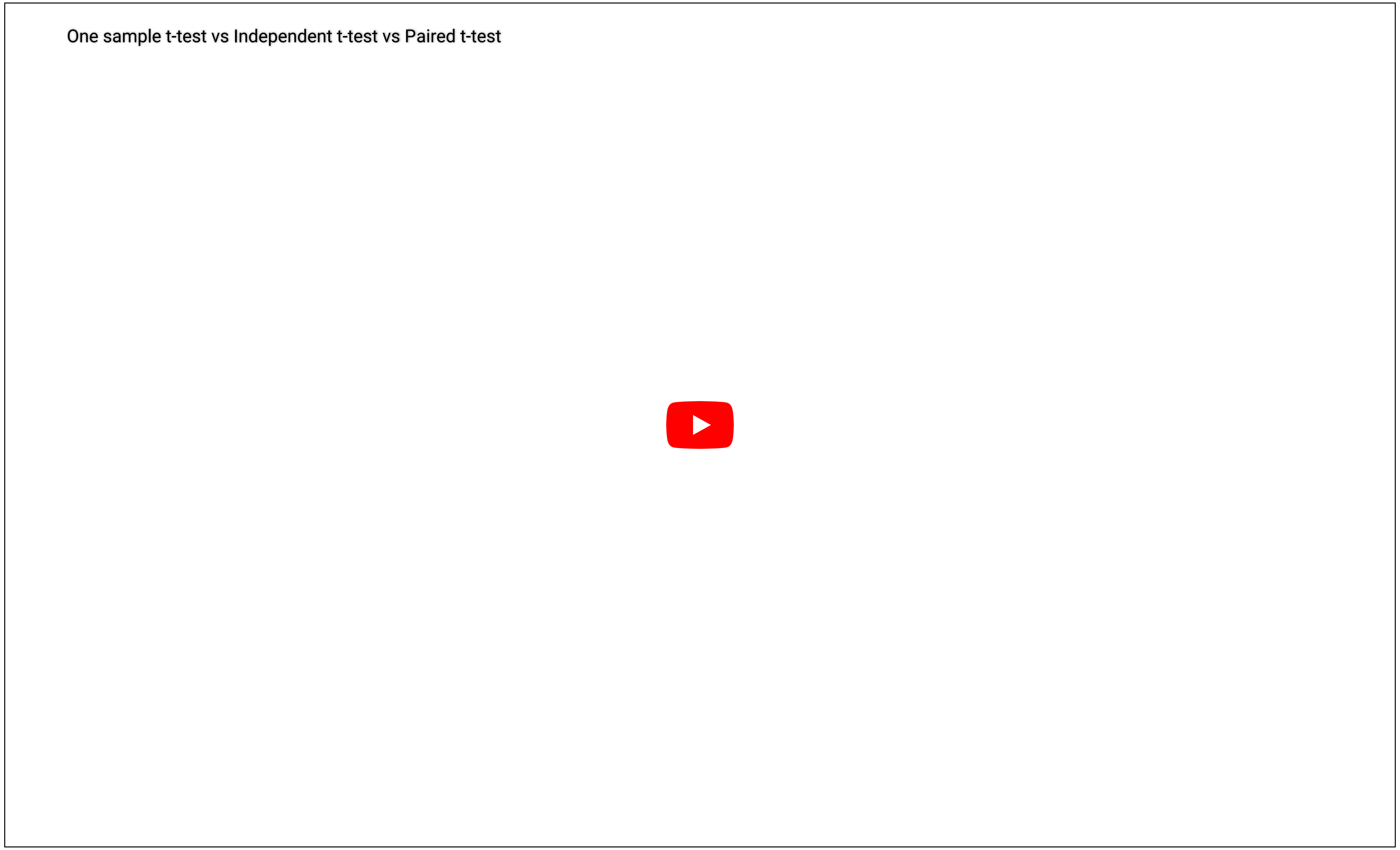
Inference for continuous data/outcomes
- One sample t-test
- 3+ independent samples
- Simple linear regression / correlation
- 2 sample tests: paired and independent
- Power and sample size
- Non-parametric tests

Inference for categorical data/outcomes
- One proportion test
- Fisher’s exact test
- Non-parametric tests
- Chi-squared test
- 2 proportion test
- Power and sample size

R
- Basics
- Reproducibility
- Quarto
- Packages
- Data visualization
- Data wrangling
- R Projects
- …

Before we get into power and sample size

Let’s watch this youtube video to jog our memory (remind us of what we learned):



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From Lesson 13: Does caffeine increase finger taps/min (on average)?

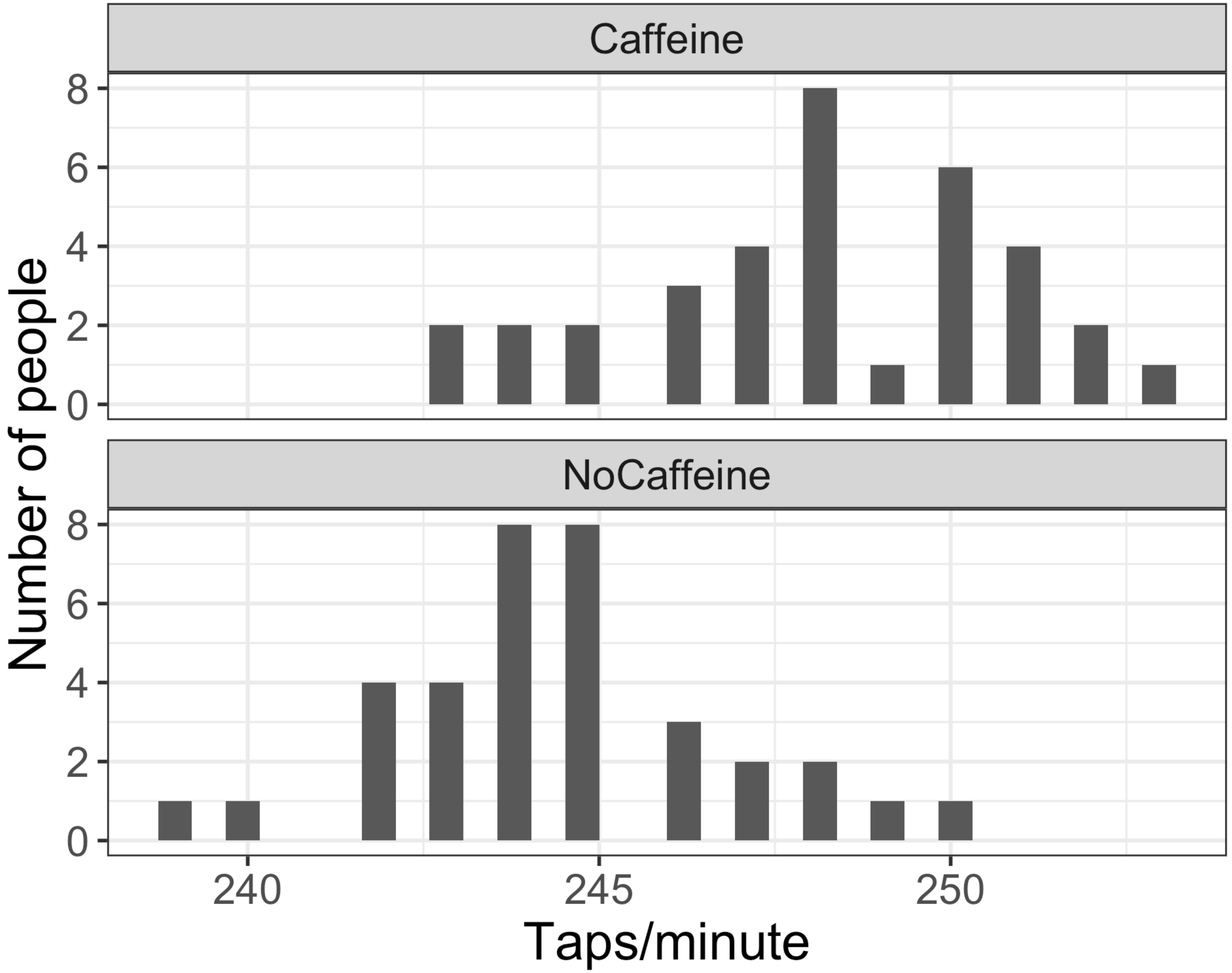
- Use this example to illustrate how to calculate a confidence interval and perform a hypothesis test for two independent samples

Study Design:[1]

- 70 college students students were trained to tap their fingers at a rapid rate
- Each then drank 2 cups of coffee (double-blind)
  - **Control** group: decaf
  - **Caffeine** group: ~ 200 mg caffeine
- After 2 hours, students were tested.
- **Taps/minute** recorded

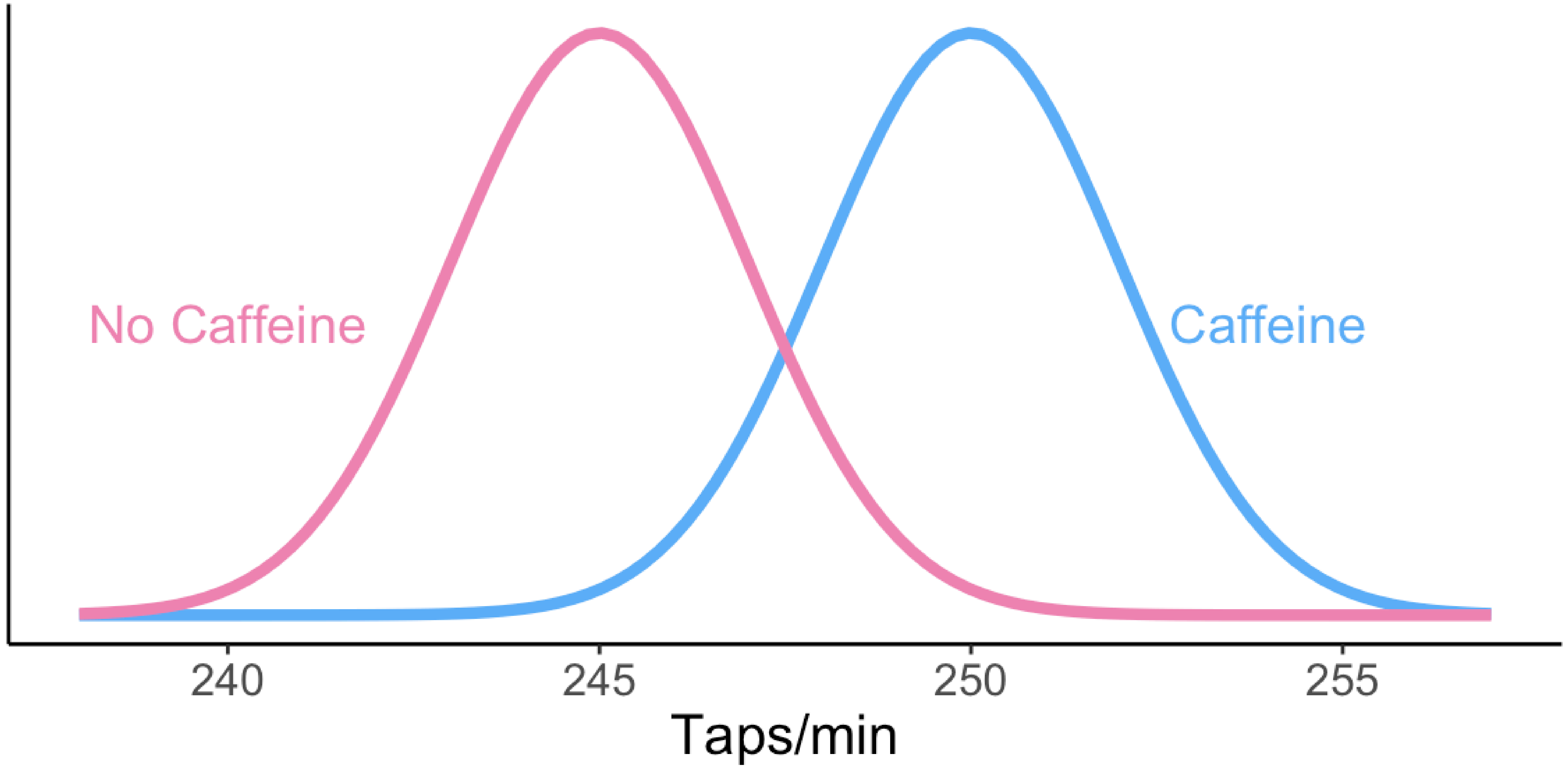
We started looking at the taps/min for each group

- Code to make these histograms



What if the following were the true population distributions? Case 1

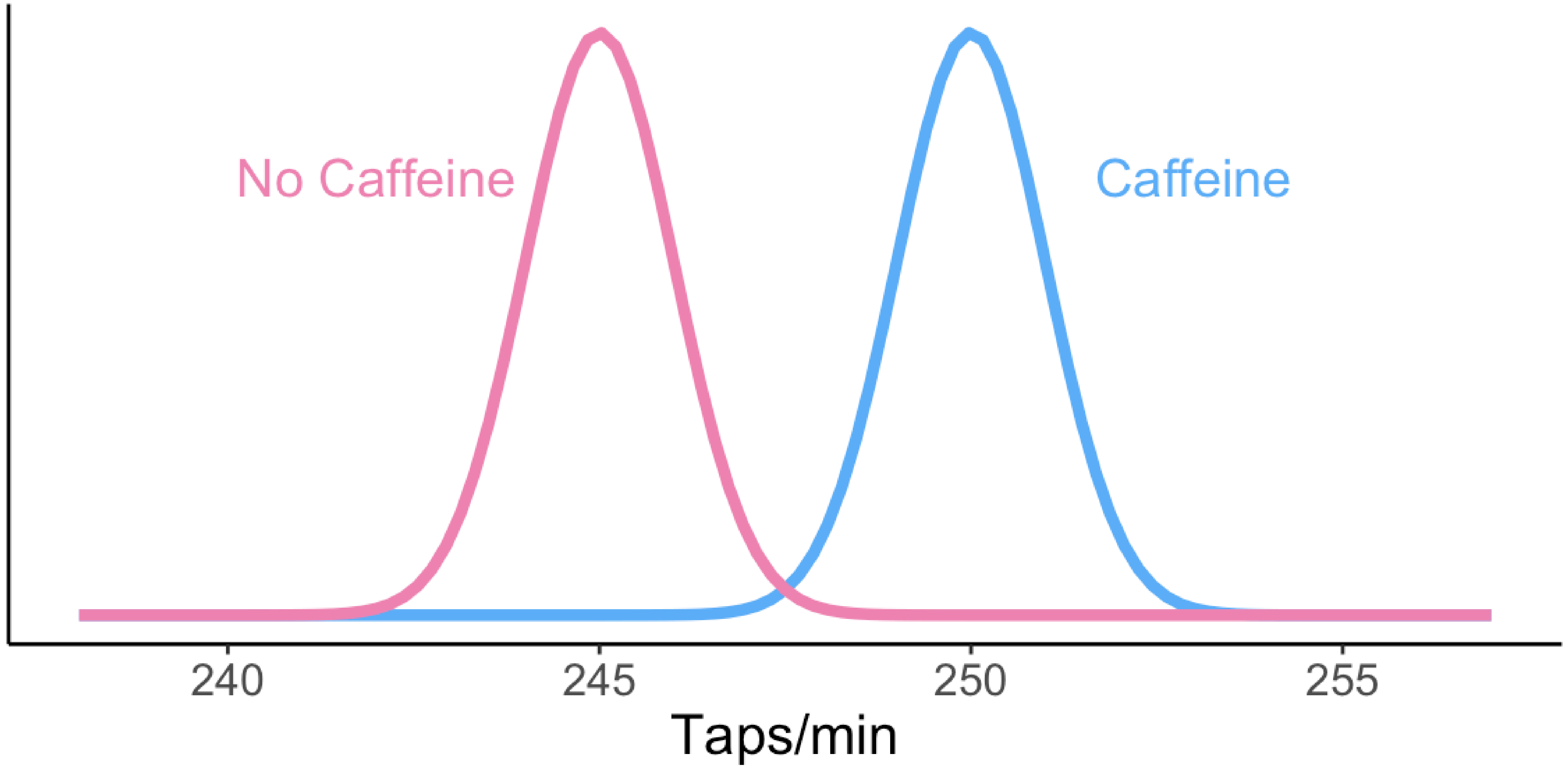
- Difference in population means is **5**
- Both have a standard deviation of 2



- When we take two samples from these groups, do you think it would be easy to distinguish between the mean taps/min?
  - **Depends on the number of samples we get: we might need a lot**

What if the following were the true population distributions? Case 2

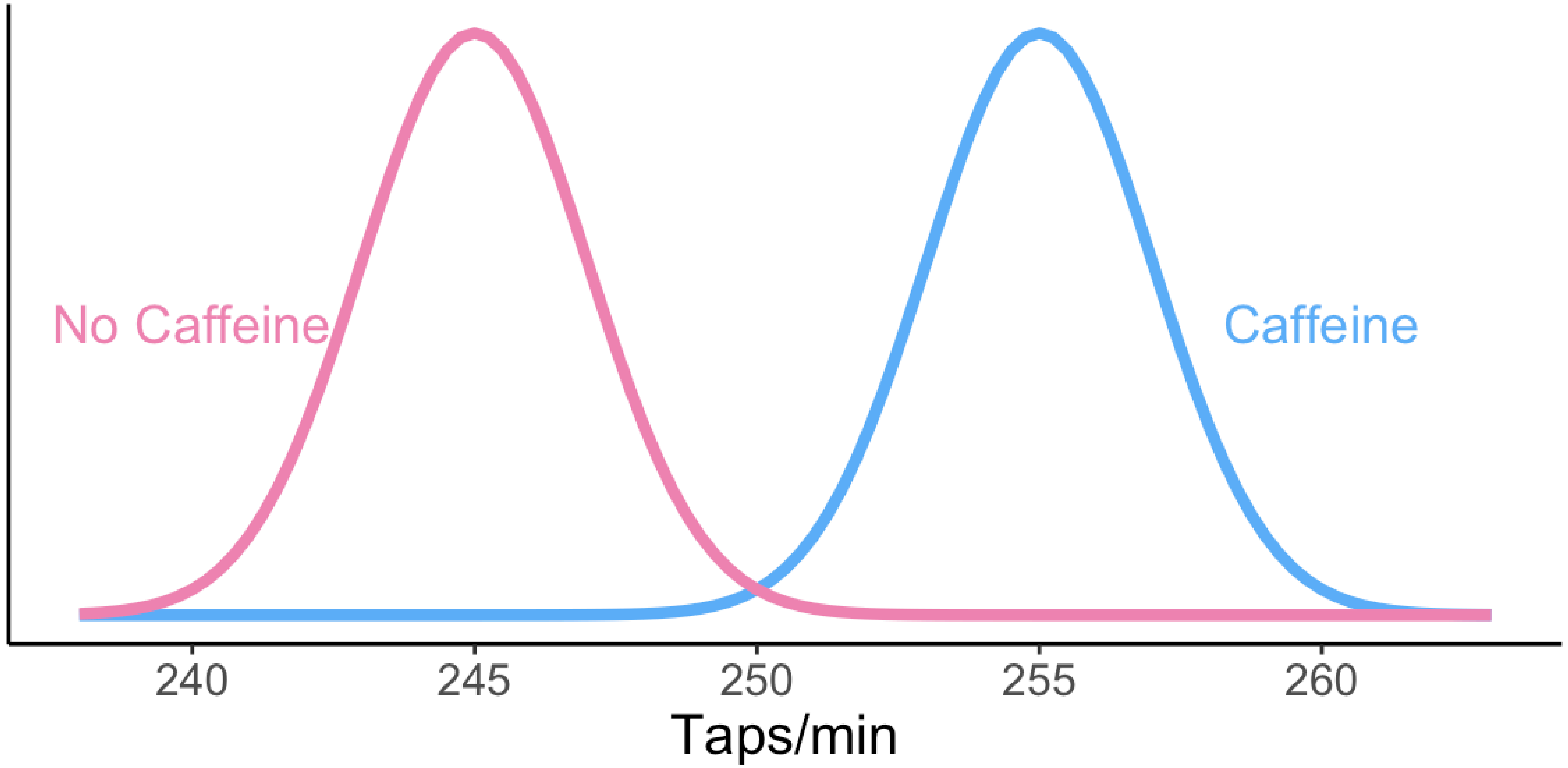
- Difference in population means is **5**
- Both have a standard deviation of 1



- When we take two samples from these groups, do you think it would be easy to distinguish between the mean taps/min?
  - **Seems easier to distinguish here.** How did the standard deviation decrease?

What if the following were the true population distributions? Case 3

- Difference in population means is **10**
- Both have a standard deviation of 2



- When we take two samples from these groups, do you think it would be easy to distinguish between the mean taps/min?
  - **Also seems easier to distinguish here**

There are a few things at play here

- There are several measurements that affect how easy it is to distinguish between two populations
- “Distinguish between two populations” = correctly reject the null hypothesis that they are the same

- What elements are at play?
  1. **Difference** in population means
  2. **Number of samples** from each population
  3. The **significance level** that we use for a cut off
  4. The **power** of our test
- More familiar with first two, but let’s define #3 and #4 more

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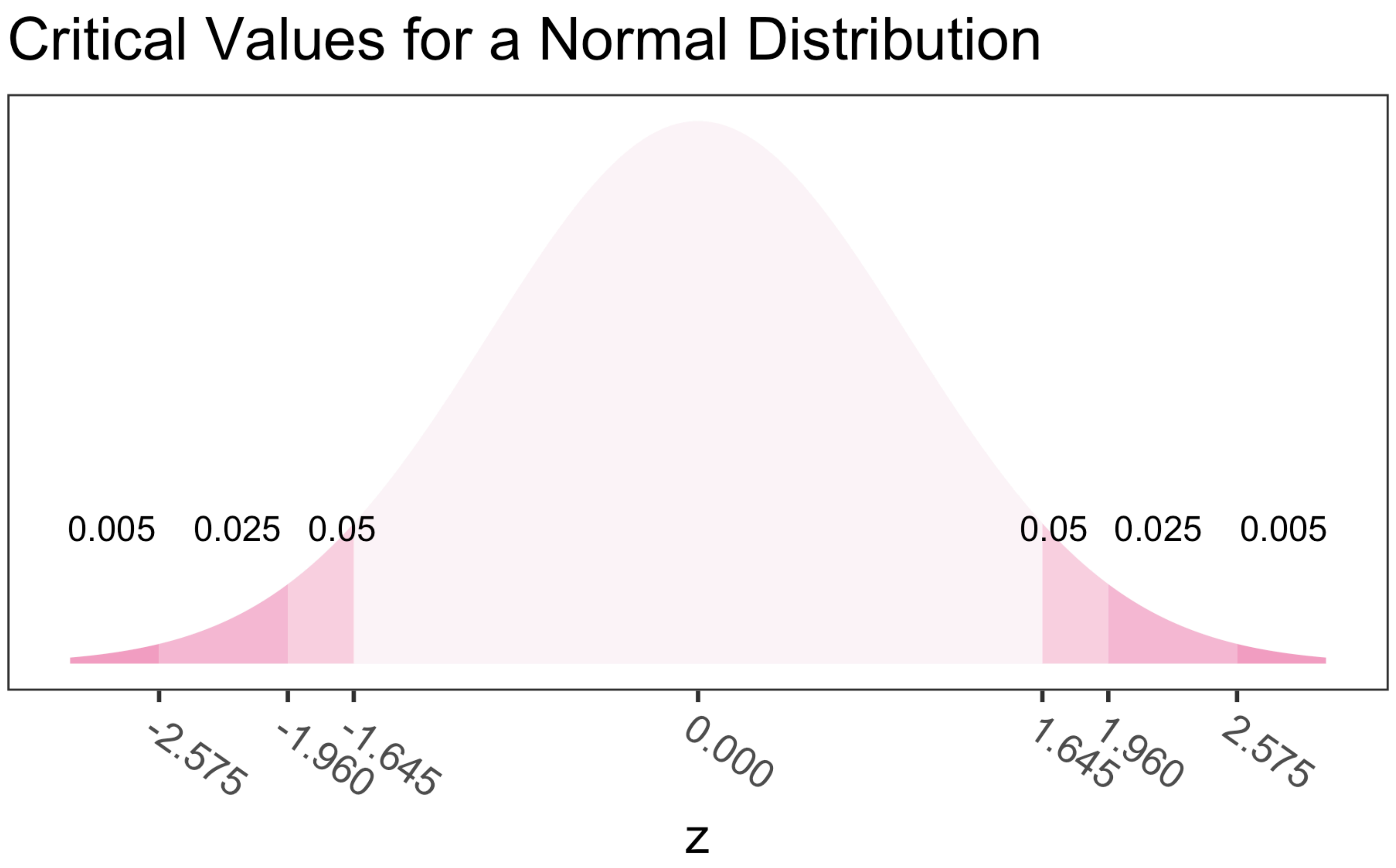
Significance levels and critical values

- **Critical values** are the cutoff values that determine whether a test statistic is statistically significant or not
  - Determined by the **significance level**
- If a test statistic is greater in absolute value than the critical value, we reject H_0



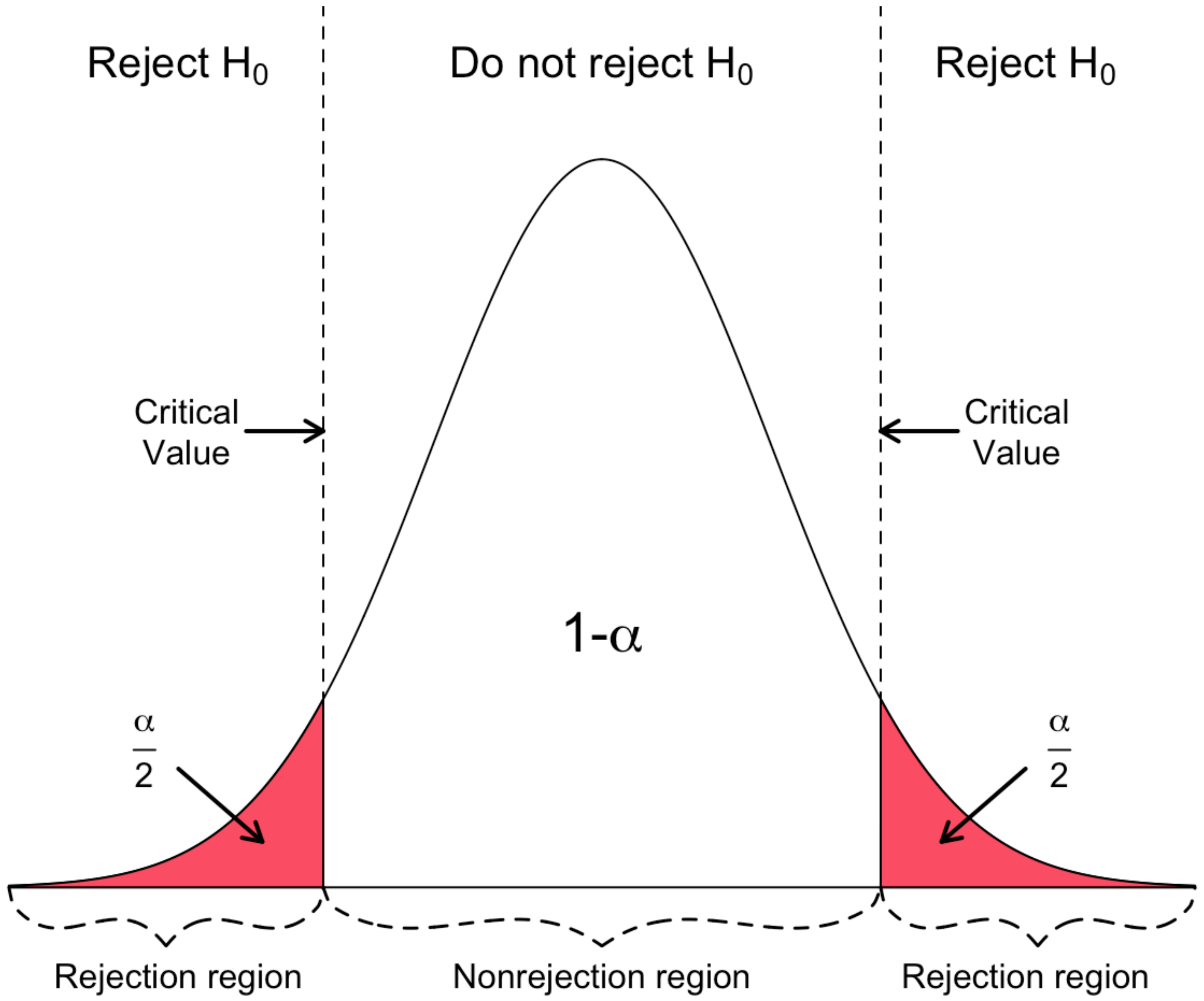
- Critical values are determined by
  - the significance level α,
  - whether a test is 1- or 2-sided, &
  - the probability distribution being used to calculate the p-value (such as normal or t-distribution)

- We have been referring to critical values from the t-distribution as t^*
  - See how we calculate a specific confidence interval in Lesson 10

Poll Everywhere Question 1

Rejection region, significance levels, and critical values

- If the absolute value of the test statistic is greater than the critical value, we reject H_0
  - In this case the test statistic is in the **rejection region**.
  - Otherwise it's in the non-rejection region.



- What do rejection regions look like for 1-sided tests?

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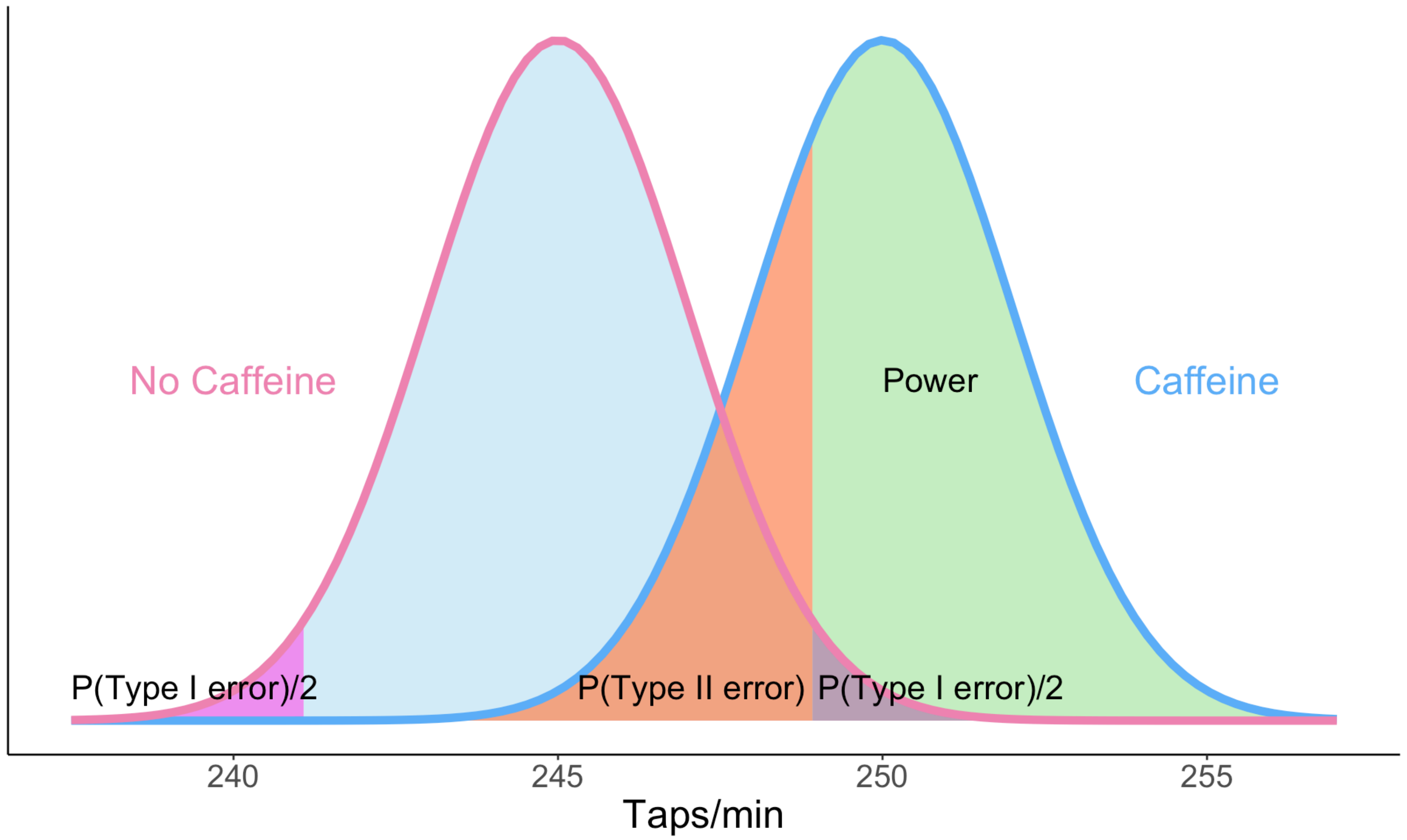
Let's start with some important definitions in words

- **Type I error** (α): Probability of rejecting the null hypothesis given that the null is true
- **Type II error** (β): Probability of failing to reject the null hypothesis given that the null hypothesis is false
- **Power** (or sensitivity) ($1-\beta$): Probability of rejecting the null hypothesis given that the null is false (correct)
- **Specificity** ($1-\alpha$): Probability of failing to reject the null hypothesis given that the null is true (correct)

	Fail to reject null hypothesis	Reject null hypothesis
Null hypothesis is true	Correct! (true negative)	Type I error (false positive) probability = α
Null hypothesis is false	Type II error (false negative) probability = β	Correct! (true positive)

What does that look like with our two populations?

	Fail to reject null hypothesis	Reject null hypothesis
Null hypothesis is true	Correct! (true negative)	Type I error (false positive) probability = α
Null hypothesis is false	Type II error (false negative) probability = β	Correct! (true positive)



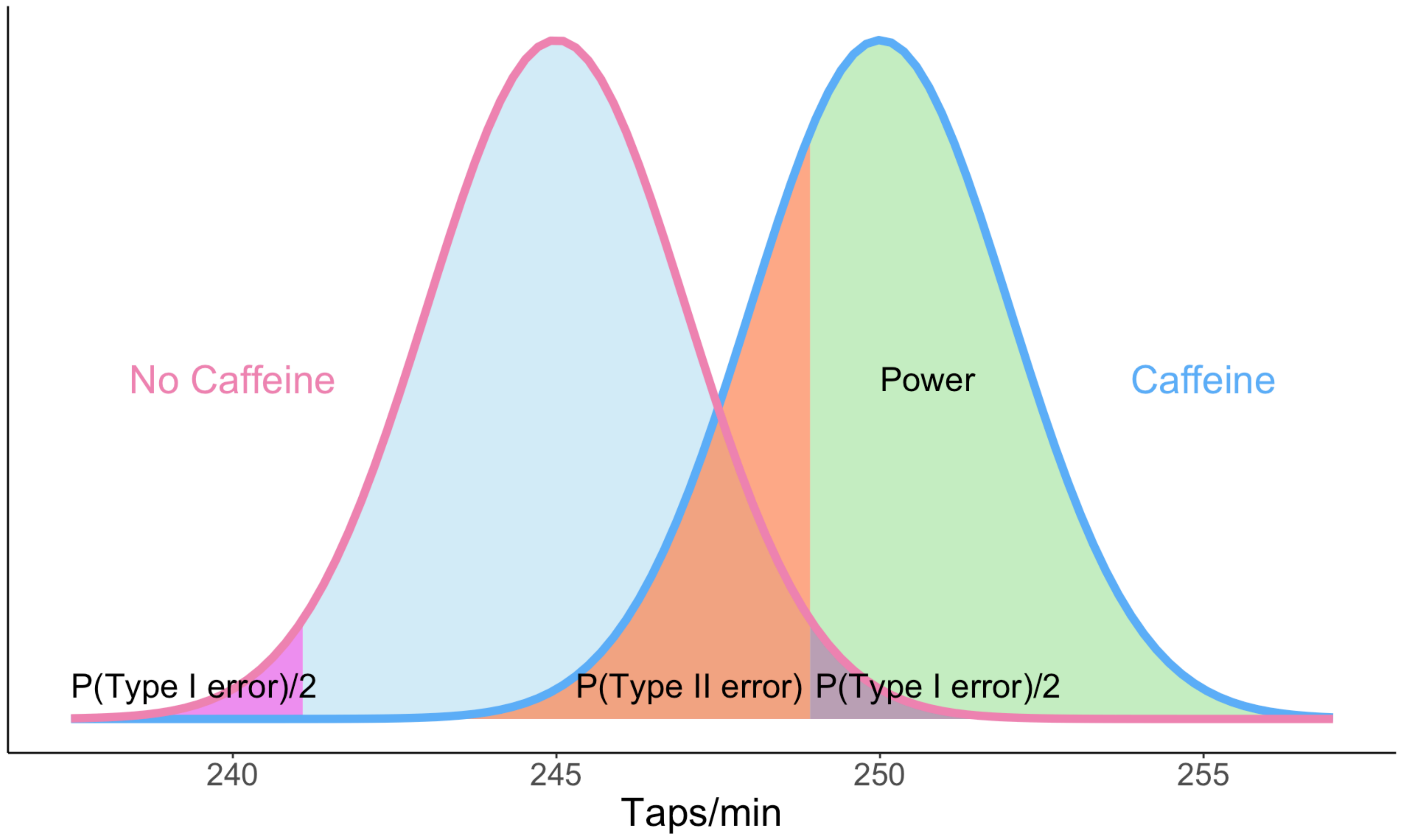
- α = probability of making a Type I error
  - This is the significance level (usually 0.05)
  - Set before study starts
- β = probability of making a Type II error
- Ideally we want
  - small Type I & II errors and
  - big power

Power (or sensitivity) ($1-\beta$): Probability of rejecting the null hypothesis given that the null is false (correct)

- Power is the correct region that is usually **in line with our study design**: studies are often seeing if there is a distinction between two populations

Power

- **Power** (or sensitivity) $(1 - \beta)$: Probability of rejecting the null hypothesis given that the null is false (correct)



- Power is also called the
  - true positive rate,
  - probability of detection, or
  - the *sensitivity* of a test

- Typically, we aim for 80% or 90% power

Let’s demonstrate the relationship between error and power

From the applet at https://rpsychologist.com/d3/NHST/

Let’s look at the following scenarios:

1. Solve for power: decreasing type 1 error (α)
2. Solve for power: increasing type 1 error (α)

- Takeaway: cannot minimize both type 1 and 2 error

3. Solve for power: decrease sample size
4. Solve for power: increase sample size

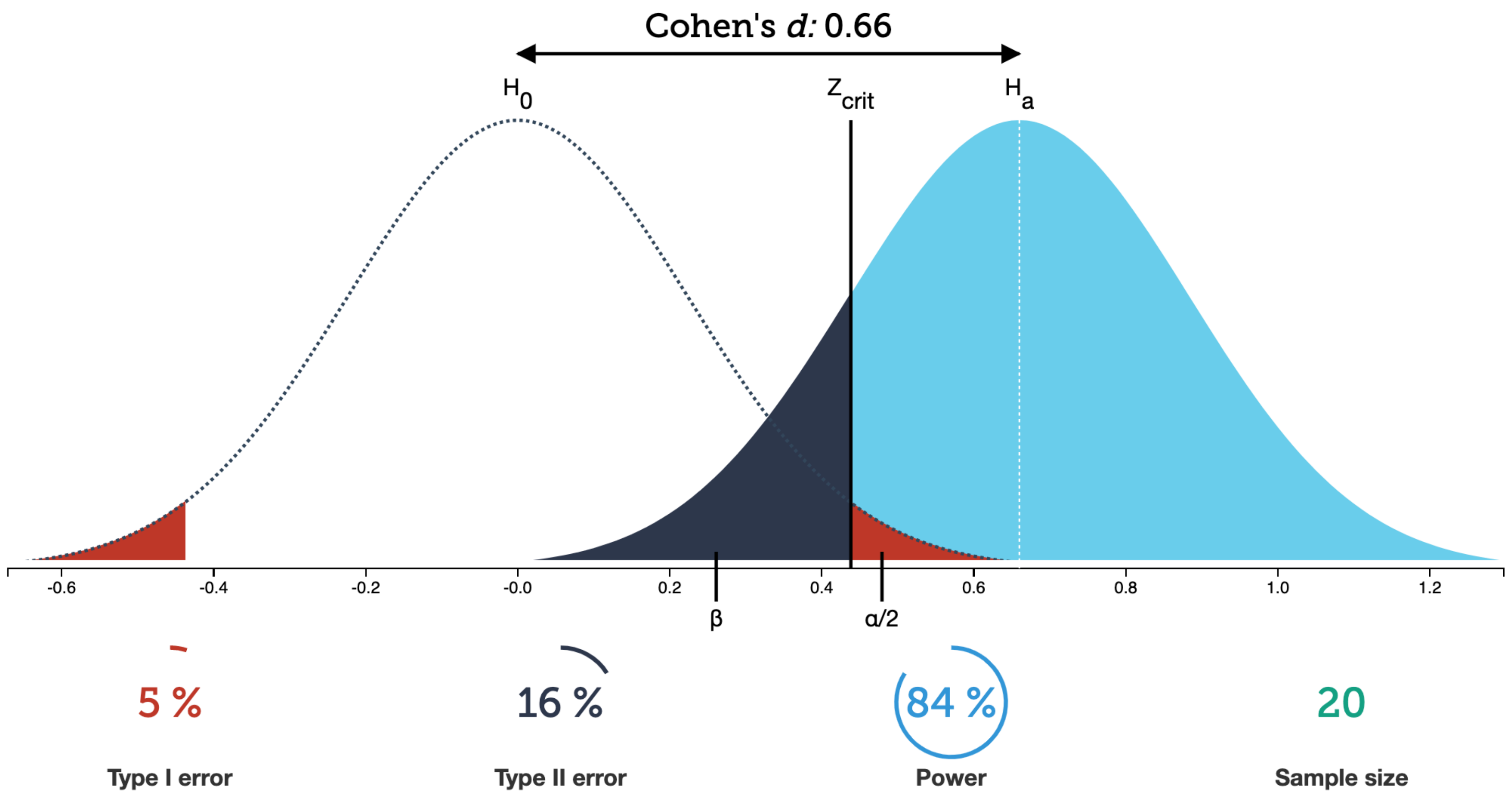
- Takeaway: increasing sample size increases power

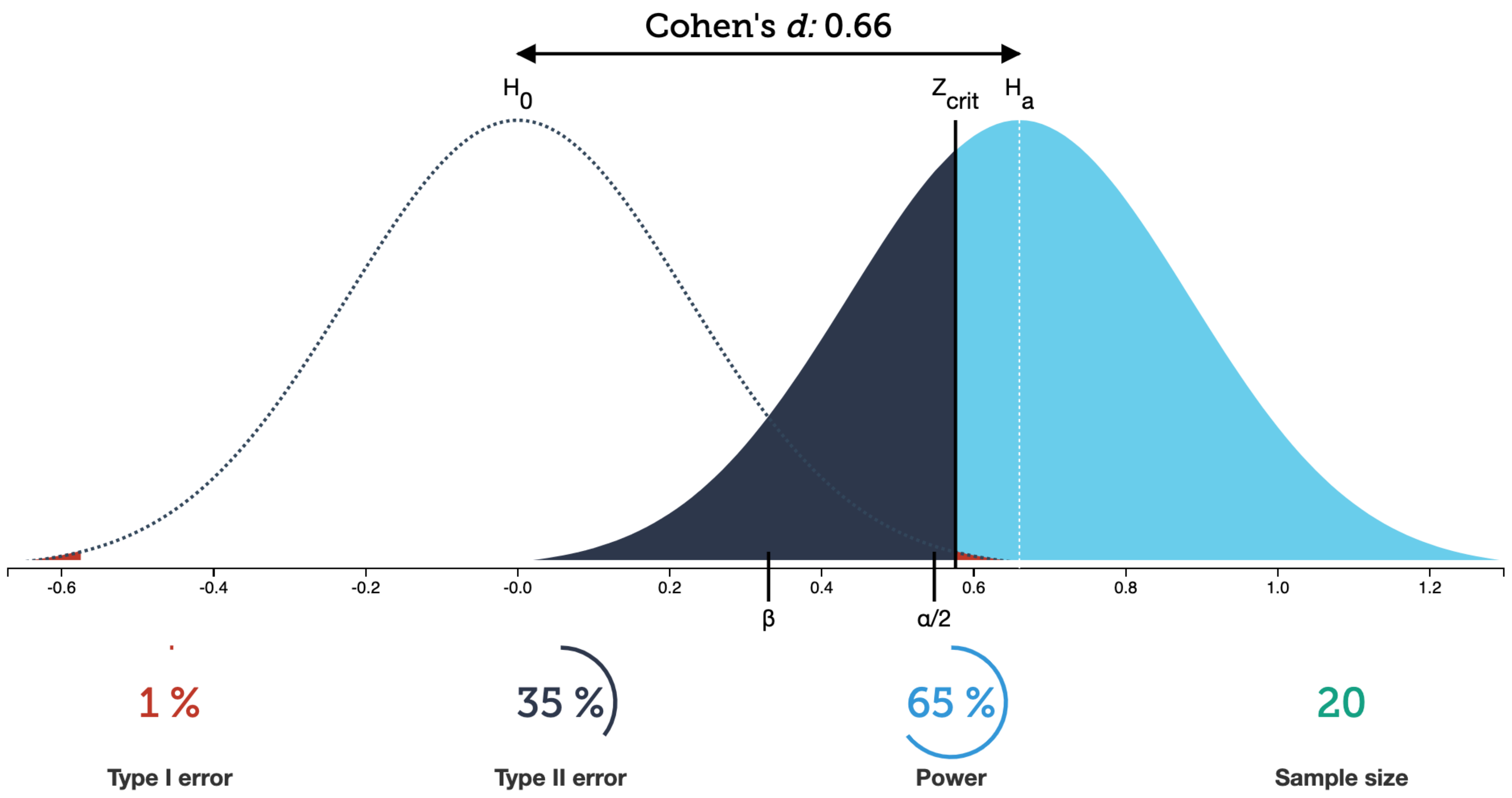
5. Solve for power: increase difference of means
6. Solve for power: decrease difference of means

- Takeaway: increasing difference in means increases power

If you want to keep revisiting these concepts!

From the applet at https://rpsychologist.com/d3/NHST/





- Cohen's d is just a stanardized value to represent the difference in means:

$$d = \frac{\overline{x}_1 - \overline{x}_2}{s}$$

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Calculating power or sample size

- Typically, before we set up a research study, we try to find the needed sample size to achieve 80% or 90% power

- If we have already have data, then we typically calculate the power based on the sample we have

Example calculating power (1/3)

Let's say we have:

- a null population with a normal distribution, centered at 0 with a standard deviation of 1 ($X_0 \sim Norm(0, 1)$)
- an alternative population, centered at 3 with a standard deviation of 1 ($X_A \sim Norm(3, 1)$)

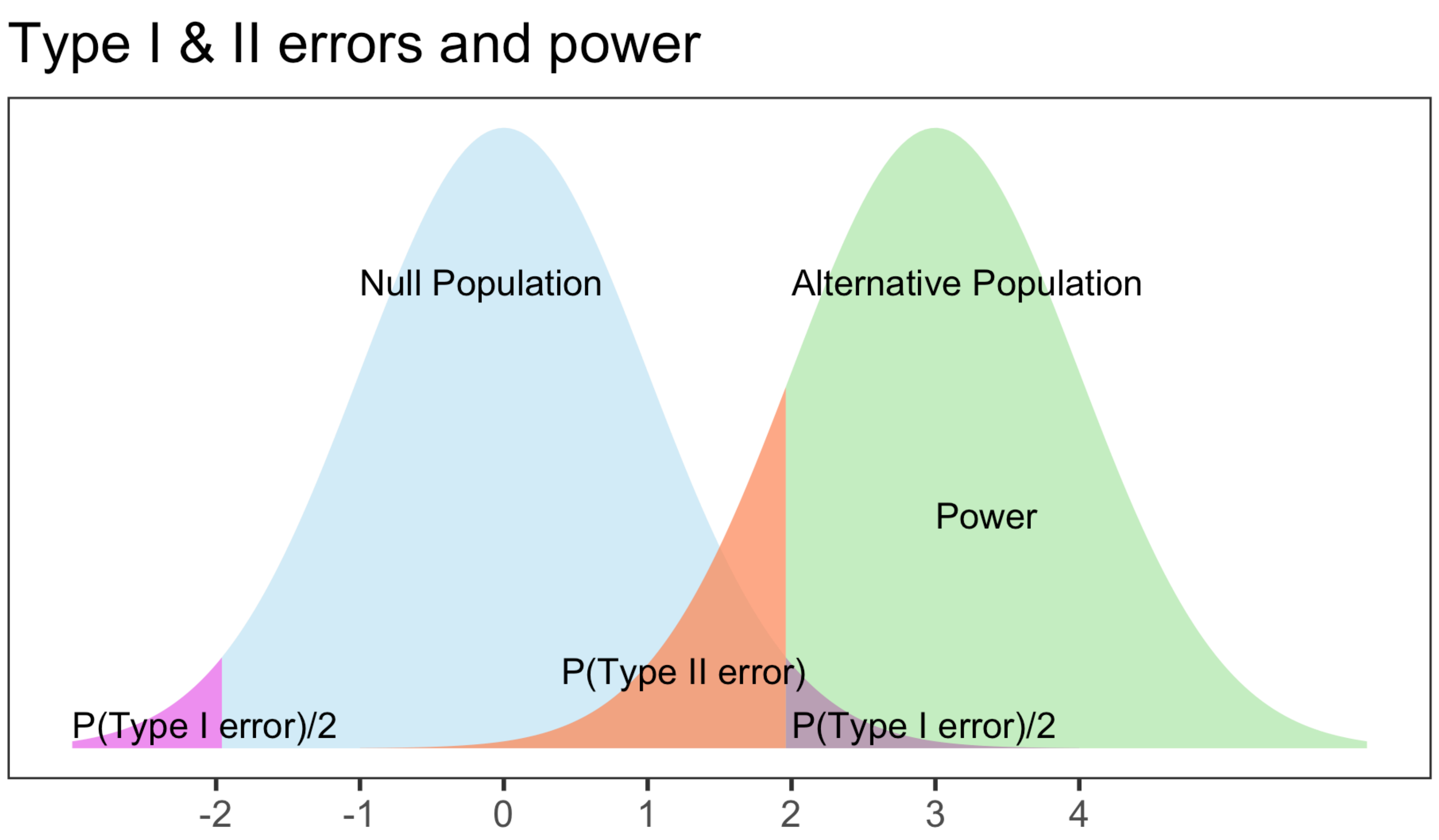
Find the power of a 2-sided test if the actual mean is 3 and our significance level is 0.05.

Example calculating power (2/3)

Let's say we have:

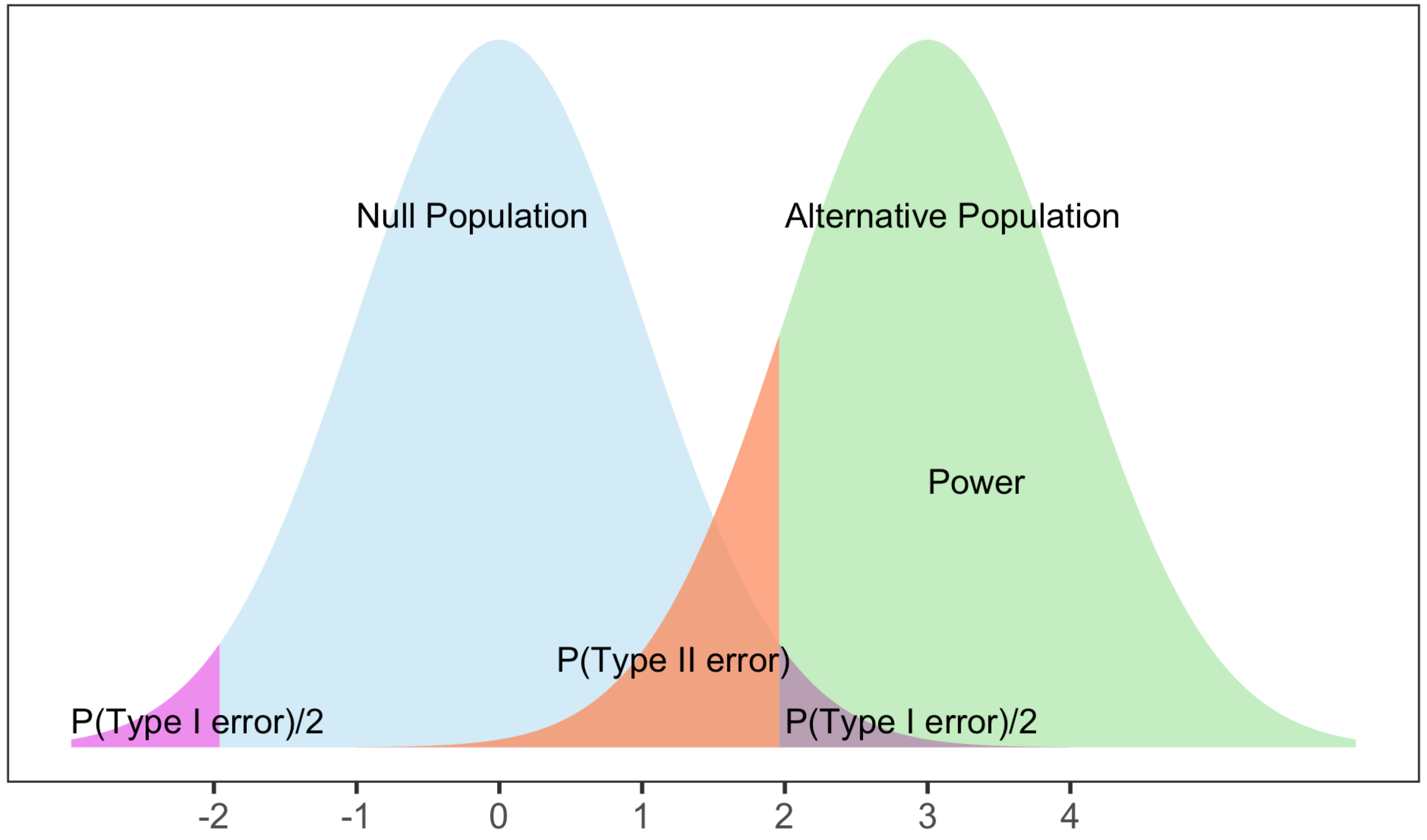
- a null population with a normal distribution, centered at 0 with a standard error of 1 ($X_0 \sim Norm(0, 1)$)
- an alternative population, centered at 3 with a standard error of 1 ($X_A \sim Norm(3, 1)$)

Find the power of a 2-sided test if the actual mean is 3 and our significance level is 0.05.



- Power = P (Reject H_0 when alternative pop is true)
  - Correctly reject null
- When α = 0.05, we reject H_0 when the test statistic z is at least 1.96 (critical value is 1.96 under the null distribution)
- Then we need to calculate the probability that we are in the rejection regions given we are **actually in the alternative population**
- Thus under the alternative population, we need to calculate $P(X_A \leq -1.96) + P(X_A \geq 1.96)$

Example calculating power (3/3)



- Thus under the alternative population, we need to calculate $P(X_A \leq -1.96) + P(X_A \geq 1.96)$
- Under the alternative population we have $X_A \sim Norm(3, 1)$

```
# left tail + right tail:
pnorm(-1.96, mean=3, sd=1,
        lower.tail=TRUE) +
  pnorm(1.96, mean=3, sd=1,
          lower.tail=FALSE)
```

```
[1] 0.8508304
```

Answer: The power is 85%

- The left tail probability `pnorm(-1.96, mean=3, sd=1, lower.tail=TRUE)` is essentially 0 in this case.
- Note that this power calculation specified the value of the SE instead of the standard deviation and sample size n individually.

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R package `pwr` for power analyses[1]

- Use `pwr.t.test` for both one- and two-sample t-tests
- Specify all parameters *except for* the one being solved for

```r
pwr.t.test(n = NULL,
           d = NULL,
           sig.level = 0.05,
           power = NULL,
           type = c("two.sample", "one.sample", "paired"),
           alternative = c("two.sided", "less", "greater"))
```

- Leave out:
  - `n`: returns sample size
  - `d`: returns Cohen's d/effect size (next slide)
  - `sig.level`: get significance level (not typical)
  - `power`: returns power

What is Cohen's d?

- d is **Cohen's d** effect size
  - Just a standardized way to measure the distance between the null mean and the alternative mean
- Examples of values: small = 0.2, medium = 0.5, large = 0.8

One-sample test (or paired t-test):

$$d = \frac{\mu - \mu_0}{s}$$

Two-sample test (independent):

$$d = \frac{\bar{x}_1 - \bar{x}_2}{s_{pooled}}$$

- $\overline{x}_1 - \overline{x}_2$ is the difference in means between the two groups that one would want to be able to detect as being significant,
- s_{pooled} is the pooled SD between the two groups - often assume have same sd in each group

Power calculation for testing one mean

Conversely, we can calculate how much power we had in our body temperature one-sample test, given the sample size of 130.

- **Calculate power**,
  - given α, n, "true" alternative mean μ, and null μ_0,
  - *assuming* the test statistic is normal (instead of t-distribution)

$$1-\beta = P\left(Z \leq z - z_{1-\alpha/2}\right) + P\left(Z \leq -z - z_{1-\alpha/2}\right) \quad , \quad \text{where } z = \frac{\mu-\mu_0}{s/\sqrt{n}}$$

Φ is the probability for a standard normal distribution

```
mu <- 98.25; mu0 <- 98.6; sd <- 0.73; alpha <- 0.05; n <- 130
(z <- (mu-mu0) / (sd/sqrt(n)) )
```

```
[1] -5.466595
```

```
(Power <- pnorm(z-qnorm(1-alpha/2)) + pnorm(-z-qnorm(1-alpha/2)))
```

```
[1] 0.9997731
```

If the population mean is 98.2 instead of 98.6, we have a 99.98% chance of correctly rejecting H_0 when the sample size is 130.

Sample size calculation for testing one mean

- Recall in our body temperature example that $\mu_0 = 98.6$ °F and $\overline{x} = 98.25$ °F.
  - The *p*-value from the hypothesis test was highly significant (very small).
  - What would the sample size n need to be for 80% power?
- **Calculate n**
  - given α, power ($1-\beta$), "true" alternative mean μ, and null μ_0
  - Calculate d: $d = \frac{\mu - \mu_0}{s}$

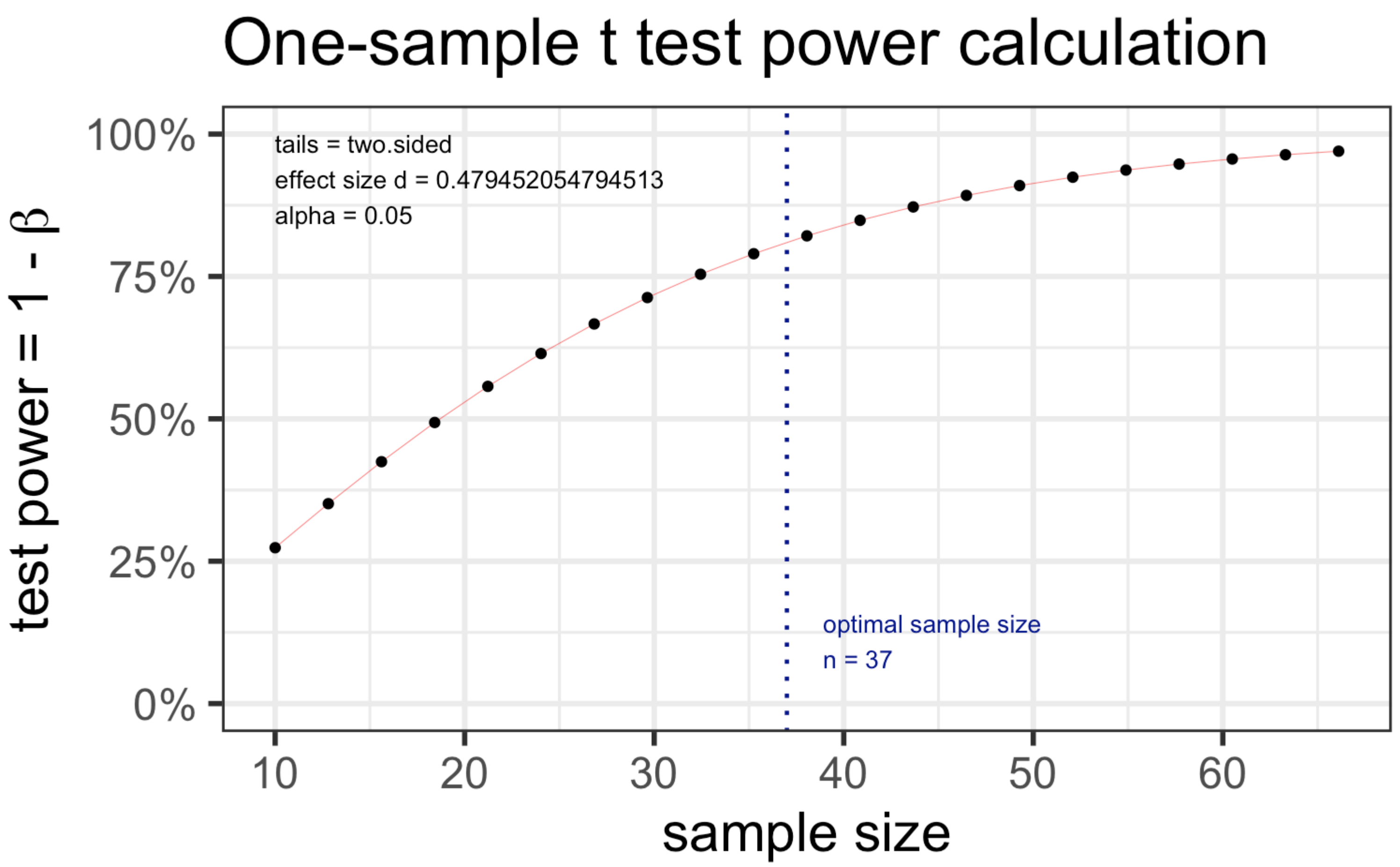
`pwr`: sample size for one mean test

Specify all parameters *except for* the sample size:

```r
library(pwr)
t.n <- pwr.t.test(
  d = (98.6-98.25)/0.73,
  sig.level = 0.05,
  power = 0.80,
  type = "one.sample")

t.n
```

```
     One-sample t test power calculation 

              n = 36.11196
              d = 0.4794521
      sig.level = 0.05
          power = 0.8
    alternative = two.sided
```

```r
plot(t.n)
```



We **need 37 individuals** to detect this difference with 80% power.

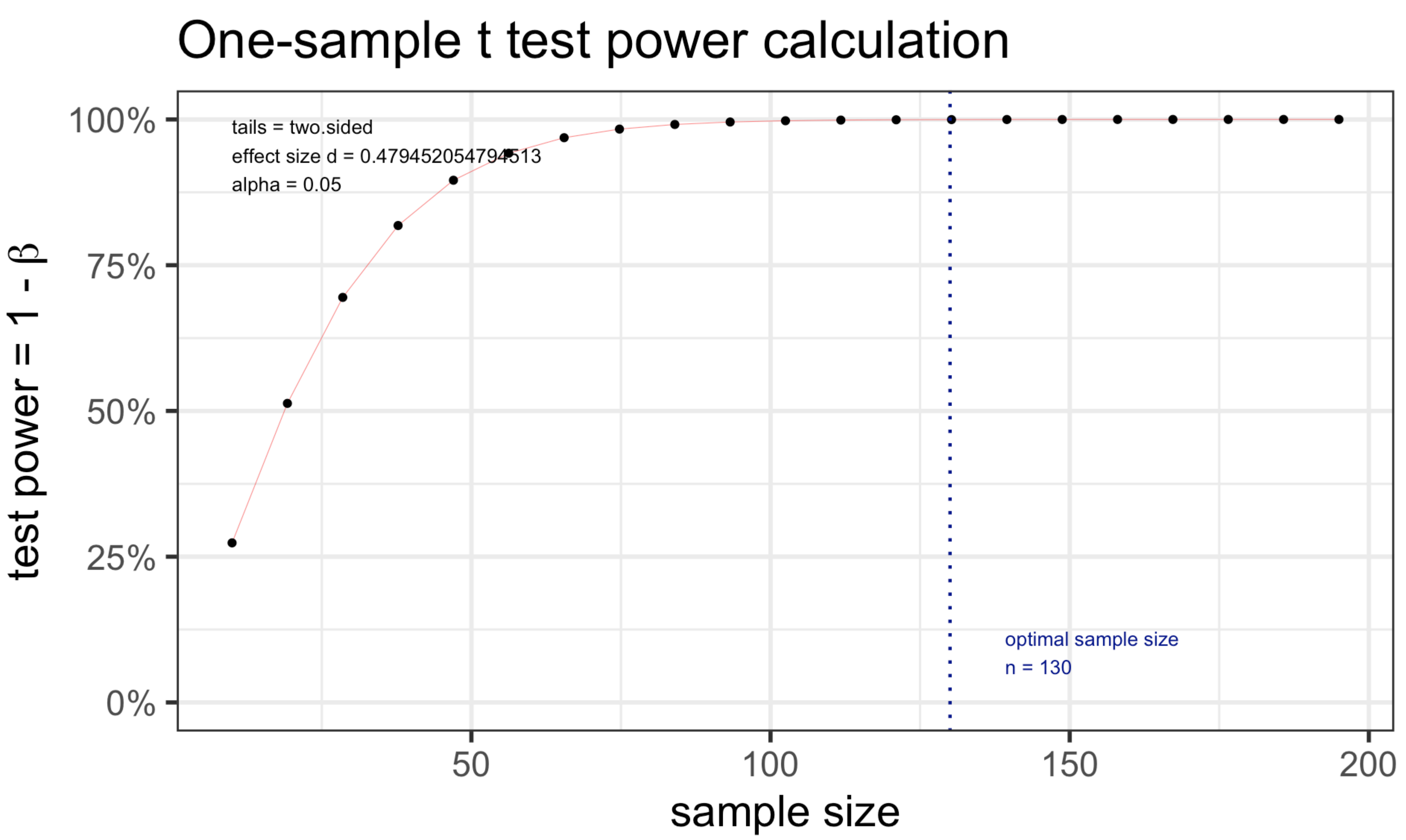
pwr: power for one mean test

Specify all parameters *except for* the power:

```
t.power <- pwr.t.test(
  d = (98.6-98.25)/0.73,
  sig.level = 0.05,
  # power = 0.80,
  n = 130,
  type = "one.sample")

t.power
```

```
     One-sample t test power
calculation

              n = 130
              d = 0.4794521
      sig.level = 0.05
          power = 0.9997354
    alternative = two.sided
```

```
plot(t.power)
```



We have **99.97% power** to detect this difference with 130 individuals.

`pwr`: Two-sample t-test: sample size

Example: Let's revisit our caffeine taps study. Investigators want to know what sample size they would need to detect a 2 point difference between the two groups. Assume the SD in both group samples is 2.6.

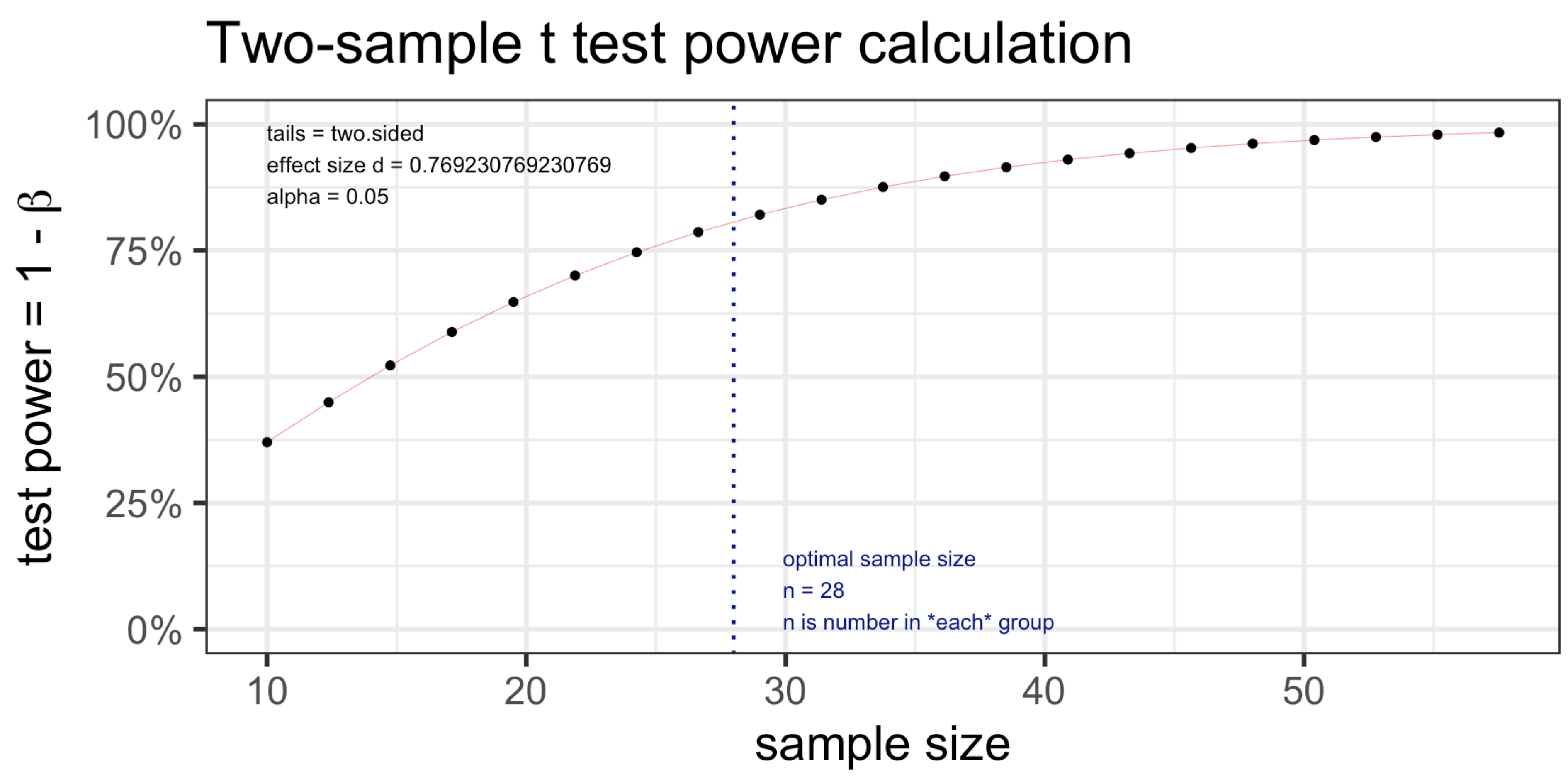
Specify all parameters *except for* the sample size:

```
t2.n <- pwr.t.test(
  d = 2/2.6,
  sig.level = 0.05,
  power = 0.80,
  type = "two.sample")

t2.n
```

```
     Two-sample t test power calculation 

              n = 27.52331
              d = 0.7692308
      sig.level = 0.05
          power = 0.8
    alternative = two.sided

NOTE: n is number in *each* group
```

::::::

```
plot(t2.n)
```



We **need 28 individuals** to detect this difference with 80% power.

`pwr`: Two-sample t-test: power

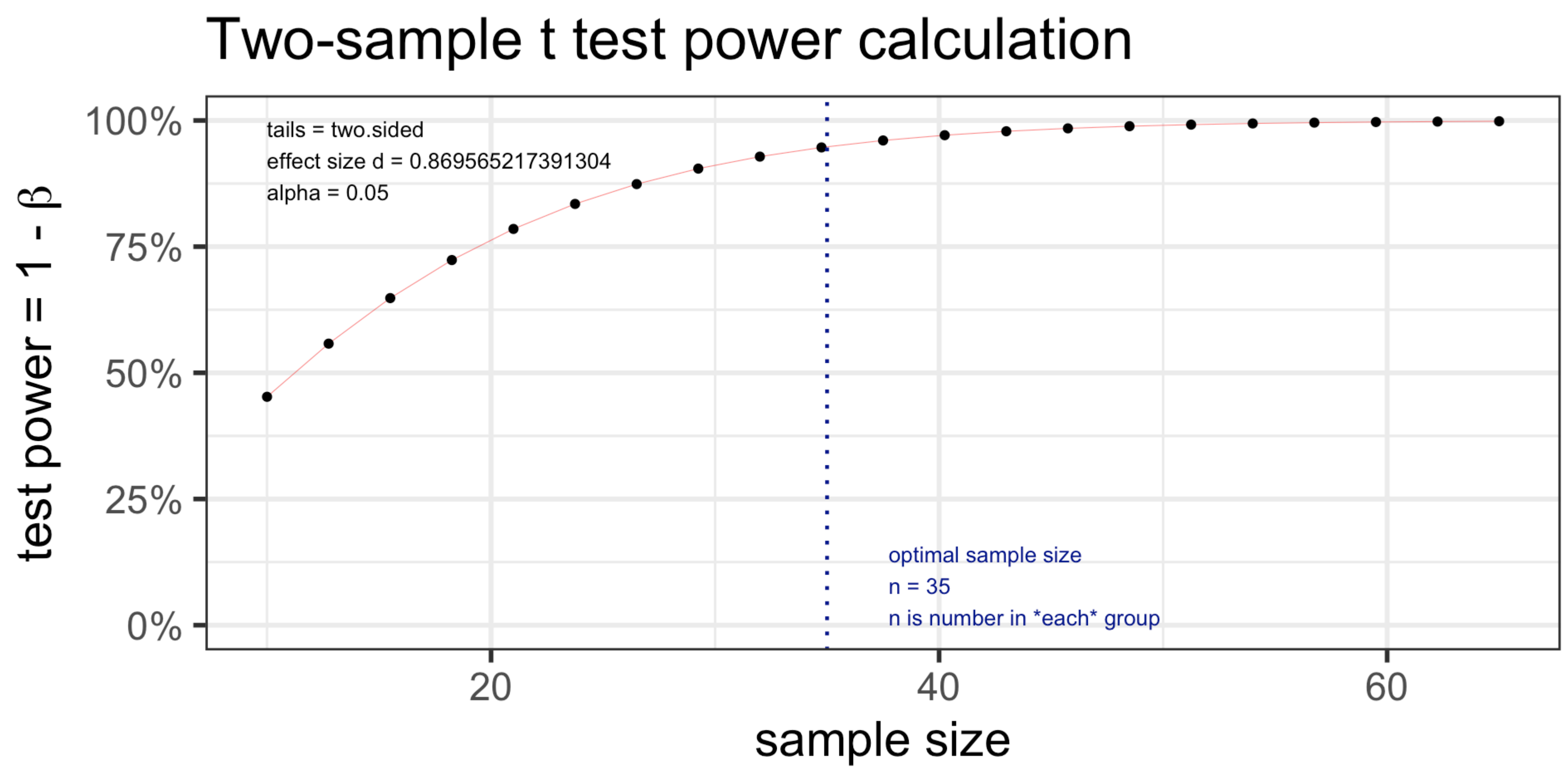
Example: Let's revisit our caffeine taps study. Investigators want to know what power they have to detect a 2 point difference between the two groups. The two groups are both size 35 (like in our previous example). Assume the SD in both group samples is 2.6.

Specify all parameters *except for* the power:

```
t2.power <- pwr.t.test(
  d = 2/2.3,
  sig.level = 0.05,
  n = 35,
  type = "two.sample")

t2.power
```

```
     Two-sample t test power calculation

              n = 35
              d = 0.8695652
      sig.level = 0.05
          power = 0.9480091
    alternative = two.sided

NOTE: n is number in *each* group
```

```
plot(t2.power)
```



We have **94.8% power** to detect this difference with 35 individuals in each group.

Resources for power and sample size calculations

More software for power and sample size calculations: PASS

- PASS is a very powerful (& expensive) software that does power and sample size calculations for many advanced statistical modeling techniques.
  - Even if you don't have access to PASS, their documentation is very good and free online.
  - Documentation includes formulas and references.
  - PASS documentation for powering means
    - One mean, paired means, two independent means
- One-sample t-test documentation: https://www.ncss.com/wp-content/themes/ncss/pdf/Procedures/PASS/One-Sample_T-Tests.pdf

OCTRI-BERD power & sample size presentations

- **Power and Sample Size 101**
  - Presented by Meike Niederhausen; April 13, 2023
  - Slides: http://bit.ly/PSS101-BERD-April2023
  - Recording
- **Power and Sample Size for Clinical Trials: An Introduction**
  - Presented by Yiyi Chen; Feb 18, 2021
  - Slides: http://bit.ly/PSS-ClinicalTrials
  - Recording
- **Planning a Study with Power and Sample Size Considerations in Mind**
  - Presented by David Yanez; May 29, 2019
  - Slides
  - Recording
- **Power and Sample Size Simulations in R**
  - Presented by Robin Baudier; Sept 21, 2023
  - Slides
  - Recording