Lesson 17: Inference for a single proportion or difference of two (independent) proportions

TB sections 8.1-8.2

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Learning Objectives

1. Remind ourselves of the Normal approximation of the binomial distribution and define the sampling distribution of a sample proportion
2. Run a hypothesis test for a single proportion and interpret the results.
3. Construct and interpret confidence intervals for a single proportion.
4. Understand how CLT applies to a difference in binomial random variables
5. Run a hypothesis test for a difference in proportions and interpret the results.
6. Construct and interpret confidence intervals for a difference in proportions.

Where are we?

Data
- Collecting data
- Categorical vs. Numeric
- Summary statistics
- Data visualization

Probability
- Probability rules
- Independence, conditional
- Random variables and probability distributions
- Linear combinations
- Binomial, Normal, and Poisson

Sampling Variability, and Statistical Inference
- Sampling distributions
- Central Limit Theorem
- Confidence Intervals
- Hypothesis tests

Inference for continuous data/outcomes
- One sample t-test
- 3+ independent samples
- Simple linear regression / correlation
- 2 sample tests: paired and independent
- Power and sample size
- Non-parametric tests

Inference for categorical data/outcomes
- One proportion test
- Fisher’s exact test
- Non-parametric tests
- Chi-squared test
- 2 proportion test
- Power and sample size

R
- Basics
- Reproducibility
- Quarto
- Packages
- Data visualization
- Data wrangling
- R Projects
- …

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Moving to categorical outcomes

- Previously, we have discussed methods of inference for numerical data
  - Our outcomes were numerical values
  - We were doing inference of **means**
  - We found confidence intervals for means
  - We ran hypothesis tests for means

- Above methods used can be extended to **categorical data**, such as binomial **proportions** or data in two-way tables

- **Categorical data arise frequently in medical research**
  - Disease outcomes and patient characteristics are often recorded in natural categories
  - **Examples:** types of treatment received, whether or not disease advanced to a later stage, or whether or not a patient responded initially to a treatment

From Lesson 5: Binomial random variable

- **One specific type of discrete random variable** is a binomial random variable

Binomial random variable

- X is a binomial random variable if it represents the number of "successes" in n independent replications (or trials) of an experiment where
  - Each replicate has two possible outcomes: either **success** or **failure**
  - The probability of success is p
  - The probability of failure is $q = 1 - p$

"distributed as"

- A binomial random variable takes on values $0, 1, 2, \ldots, n$.
- If a r.v. X is modeled by a Binomial distribution, then we write in shorthand $X \sim \text{Binom}(n, p)$
- Quick example: The number of heads in 3 tosses of a fair coin is a binomial random variable with parameters $n = 3$ and $p = 0.5$.

From Lesson 5: Binomial distribution

Distribution of a **Binomial** random variable

Let X be the total number of successes in n independent trials, each with probability p of a success. Then probability of observing exactly k successes in n independent trials is

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, 2, \ldots, n$$

- The parameters of a binomial distribution are p and n.
- If a r.v. X is modeled by a binomial distribution, then we write in shorthand $X \sim \text{Binom}(n, p)$

Mean and variance of a Binomial r.v

If X is a binomial r.v. with probability of success p, then $E(X) = np$ and $\text{Var}(X) = np(1-p)$

From Lesson 6: Normal Approximation of the Binomial Distribution

- Also known as: **Sampling distribution of $\widehat{p}$** sample proportions
- If $X \sim \text{Binomial}(n, p)$ and $np > 10$ and $nq = n(1-p) > 10$
  - Ensures sample size (n) is moderately large and the p is not too close to 0 or 1
  - Other resources use other criteria (like $npq > 5$ or $np > 5$)

$E(X)$ $\sqrt{Var(X)}$

- THEN approximately

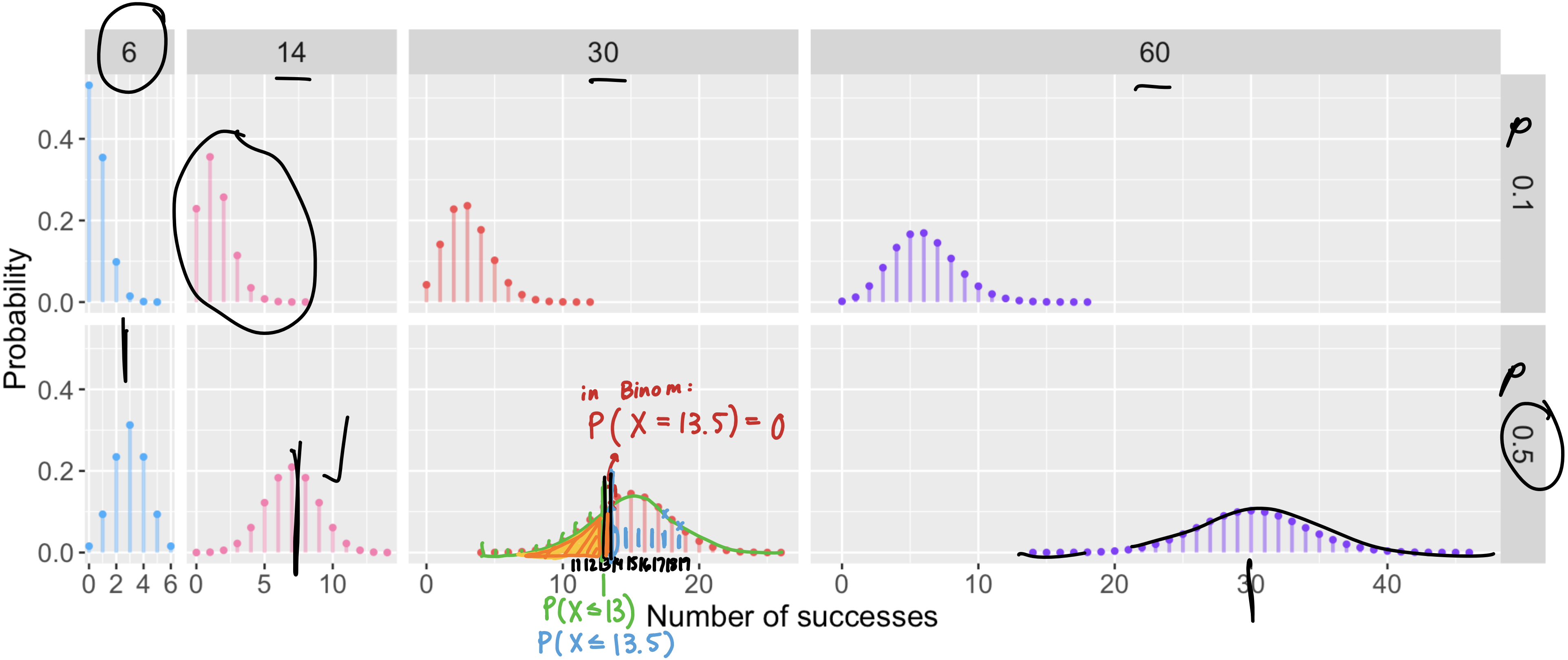
$X \sim Norm(np, \sqrt{np(1-p)})$

$$X \sim \text{Normal}(\mu_X = np, \sigma_X = \sqrt{np(1-p)})$$

- **Continuity Correction**: Applied to account for the fact that the binomial distribution is discrete, while the normal distribution is continuous
  - Adjust the binomial value (# of successes) by ±0.5 before calculating the normal probability.
  - For $P(X \leq k)$ (Binomial), you would instead calculate $P(X \leq k + 0.5)$ (Normal approx)
  - For $P(X \geq k)$ (Binomial), you would instead calculate $P(X \geq k - 0.5)$ (Normal approx)

We can look at a plot of Binomial distributions

- Binomial distributions for different n (columns) and p (rows)



Poll Everywhere Question 1

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Under which conditions can the normal approximation of the binomial distribution be used?

When $n > 30$ regardless of p	0%
When $np > 10$ and $n(1-p) > 10$	100%
When $n > 20$ and $p \approx \dots$	0%
Only when $n > 50$ and p is known	0%

$np(1-p) > 10$

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Sampling distribution of $\hat{p}$

- $\hat{p} = \frac{X}{n}$ where X is the number of "successes" and n is the sample size.
- $X \sim Bin(n, p)$, where p is the population proportion.
- For n "big enough", the normal distribution can be used to approximate a binomial distribution:

$$X \sim N\left(\mu = np, \sigma = \sqrt{np(1-p)}\right)$$

- Since $\hat{p} = \frac{X}{n}$ is a linear transformation of X, we have for large n:

$$E(\hat{p}) = E\left(\frac{X}{n}\right) = \frac{1}{n} \cdot E(X) = \frac{1}{n} np = p$$

$$\hat{p} \sim N\left(\mu_{\hat{p}} = p, \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}\right)$$

$$Var(\hat{p}) = Var\left(\frac{X}{n}\right) = \left(\frac{1}{n}\right)^2 Var(X)$$

- What is "big enough"? At least 10 successes and 10 failures are expected in the sample: $np \geq 10$ and $n(1-p) \geq 10$

For proportions: Population parameters vs. sample statistics

Population parameter

- Proportion: p, π ("pi")

↳ underlying truth

p_0 : prescribed value for p (something like 0.6)

placeholder for an actual #

while p is not a placeholder for a #

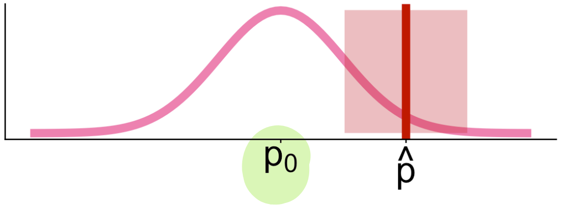
Sample statistic (point estimate)

- Sample proportion: $\hat{p}$ ("p-hat")

Approaches to answer a research question

- **Research question is a generic form for a single proportion:** Is there evidence to support that the population proportion is different than p_0?

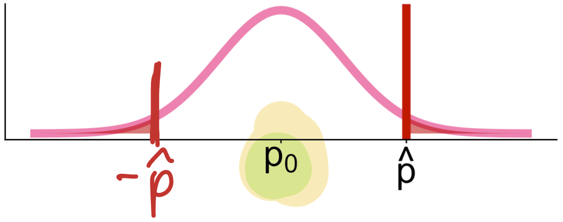
Calculate **CI for the proportion** p:



$$\hat{p} \pm z^* \cdot SE_{\hat{p}} = \hat{p} \pm z^* \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- with z^* = z-score that aligns with specific confidence interval

Run a **hypothesis test:**



Hypotheses

$$H_0: p = p_0$$
$$H_A: p \neq p_0$$
$$(or\ <, >)$$

Test statistic

$$z_{\hat{p}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 \cdot (1-p_0)}{n}}}$$

do NOT use $\hat{p}$ for SE

R code: 1- and 2-sample proportions tests

```r
prop.test(x,
          n,
          p = NULL,
          alternative = c("two.sided", "less", "greater"),
          conf.level = 0.95,
          correct = TRUE)
```

- x: Counts of successes (can have one x or a vector of multiple x's)
- n: Number of trails (can have one n or a vector of multiple n's)

R will calc $\hat{p} = \frac{x}{n}$

- p: Null value that we think the population proportion is aka p_0
- alternative: If alternative hypothesis is $\neq$, $<$, or $>$
  - Default is "two.sided" ($\neq$)
- conf.level = Confidence level $(1-\alpha)$
  - Default is 0.05
- correct: Continuity correction, whether we should use it or not

  True

  False
  - Default is TRUE (Nicky says keep it this way!)

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Example: immune response to advanced melanoma

- Looking for therapies that trigger an immune response to advanced melanoma
- In a study where 52 patients were **treated concurrently with two new therapies**, nivolumab and ipilimumab
  - 21 had an immune response.[1]
- **Outcome:** whether or not each person has an immune response

Questions that can be addressed with inference…

- What is the **estimated population** proportion **probability of immune response** following concurrent therapy with nivolumab and ipilimumab? (calculate $\hat{p}$)
- What is the **95% confidence interval for the estimated population** proportion **probability of immune response** following concurrent therapy with nivolumab and ipilimumab? (95% CI of p)
- In previous studies, the proportion of patients responding to one of these agents was 30% or less. Do these results suggest that the **probability of response to concurrent therapy is better than 0.30**? (Hypothesis test of null of 0.3)

Reference: Steps in a Hypothesis Test

1. Check the **assumptions** ✓
2. Set the **level of significance** α ✓
3. Specify the **null** (H_0) and **alternative** (H_A) **hypotheses** ✓
    1. In symbols
    2. In words
    3. Alternative: one- or two-sided?

before we get into data

4. Calculate the **test statistic**.
5. Calculate the **p-value** based on the observed test statistic and its sampling distribution
6. Write a **conclusion** to the hypothesis test
    1. Do we reject or fail to reject H_0?
    2. Write a conclusion in the context of the problem

Step 1: Check the assumptions (easier to do after Step 3)

The sampling distribution of $\hat{p}$ is approximately normal when

1. The sample observations are independent, and
2. At least 10 successes and 10 failures are expected in the sample: $np_0 \geq 10$ and $n(1-p_0) \geq 10$.

expected # successes

expected # failures

- Since p is unknown, it is necessary to substitute p_0 (the null value) for p when using the standard error to conduct hypothesis tests
  - Because we are assuming the standard error of the null hypothesis!

we will not use $\hat{p}$

- For the example, we have $p_0 = 0.30$ (not $\hat{p}$)
  - We check: $np_0 = 52 \cdot 0.3 = 15.6 > 10$
  - We check: $n(1-p_0) = 52(1-0.3) = 36.4 > 10$

✓ assumptions met, use Normal approx.

Step 2: Set the level of significance

- **Before doing a hypothesis test**, we set a cut-off for how small the p-value should be in order to reject H_0.
- Typically choose $\alpha = 0.05$

12

- See Lesson ~~11~~: Hypothesis Testing 1: Single-sample mean

Step 3: Null & Alternative Hypotheses (1/2)

Notation for hypotheses (for paired data)

$$H_0 : p = p_0$$
$$\text{vs. } H_A : p \neq, <, \text{or}, > p_0$$

Hypotheses test for example

$$H_0 : p = 0.30$$
$$\text{vs. } H_A : p \neq 0.30$$

We call p_0 the *null value* (hypothesized population mean difference from H_0)

$H_A : p \neq p_0$

- not choosing a priori whether we believe the population proportion is greater or less than the null value p_0

$H_A : p < p_0$

- believe the population proportion is **less** than the null value p_0

$H_A : p > p_0$

- believe the population population proportion is **greater** than the null value p_0

- $H_A : p \neq p_0$ is the most common option, since it's the most conservative

Step 3: Null & Alternative Hypotheses (2/2)

Null and alternative hypotheses in **words** and in **symbols**.

One sample test

- H_0: For individuals who have advanced melanoma and received a treatment of nivolumab and ipilimumab, the population proportion of immune response is 0.30
- H_A: For individuals who have advanced melanoma and received a treatment of nivolumab and ipilimumab, the population proportion of immune response is NOT 0.30

$$\begin{aligned} H_0 &: p = 0.30 \\ H_A &: p \neq 0.30 \end{aligned}$$

Step 4: Test statistic

$$\hat{p} \sim N\left(0.3, \sqrt{\frac{0.3(0.7)}{52}}\right)$$

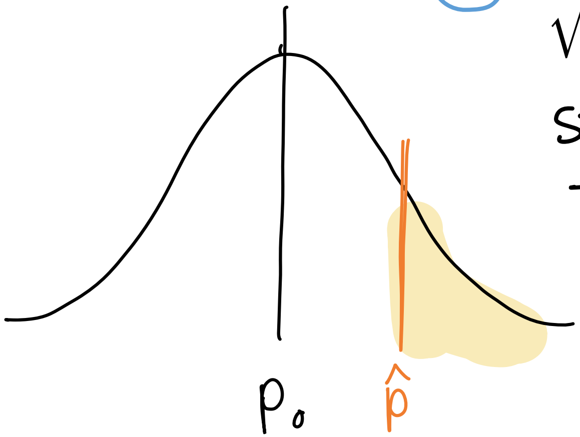
Sampling distribution of $\hat{p}$ if we assume $H_0 : p = p_0$ is true:

$$\hat{p} \sim N\left(\mu_{\hat{p}} = p, \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}\right) \sim N\left(\mu_{\hat{p}} = p_0, \sigma_{\hat{p}} = \sqrt{\frac{p_0 \cdot (1-p_0)}{n}}\right)$$

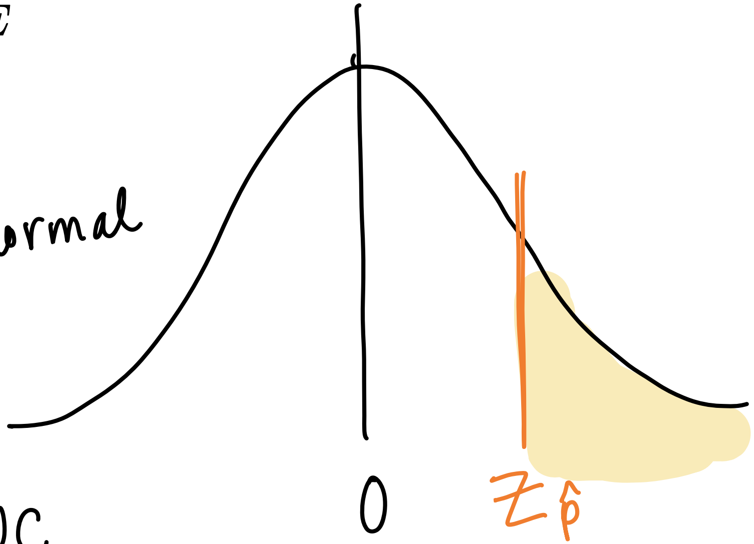
Test statistic for a one sample proportion test:

$$\text{test stat} = \frac{\text{point estimate} - \text{null value}}{SE}$$

$$z_{\hat{p}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 \cdot (1-p_0)}{n}}}$$



standard normal →



same AUC

Step 4: Test statistic

52 dual trt

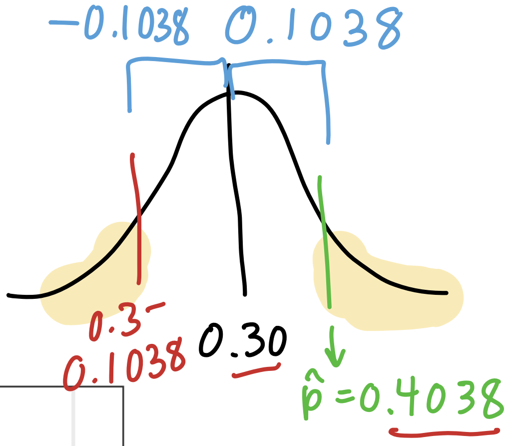
21 immune response

From our example: Recall that $\hat{p} = \frac{21}{52} = 0.4038$, $n = 52$, and $p_0 = 0.30$

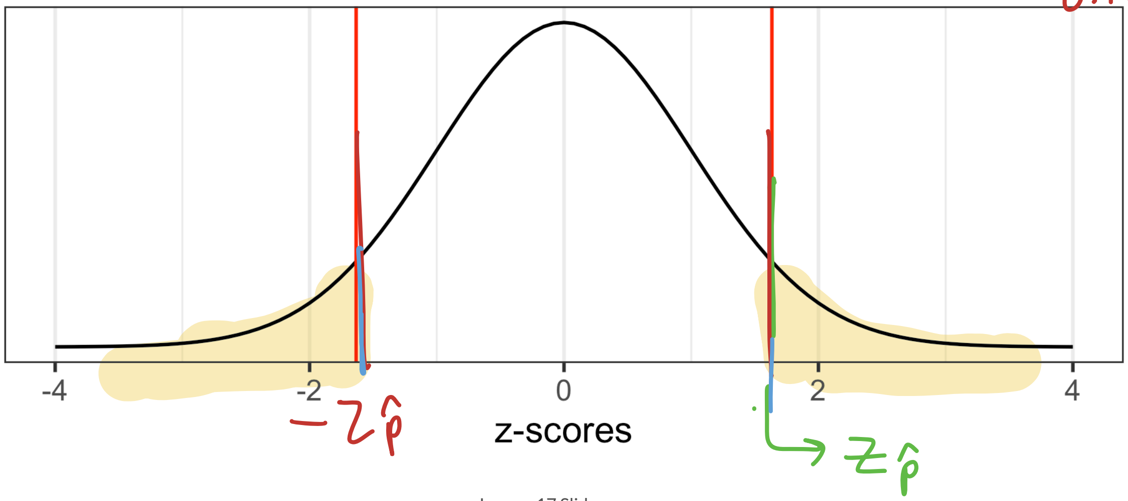
INSAMPLE

The test statistic is:

$$z_{\hat{p}} = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0\cdot(1-p_0)}{n}}} = \frac{21/52 - 0.30}{\sqrt{\frac{0.30\cdot(1-0.30)}{52}}} = 1.6341143$$



- Let's see the z-score on a Z-distribution (Standard Normal curve)



Poll Everywhere Question 2

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Based on the test statistic of 1.634 (and our $\alpha = 0.05$ and our two-si… the rejection region?

Yes, because 1.634 is greater than our critical value (z^*)	19%
Yes, because 1.634 is less than our critical value (z^*)	31%
No, because 1.634 is greater than our critical value (z^*)	6%
No, because 1.634 is less than our critical value (z^*) ✓	44%

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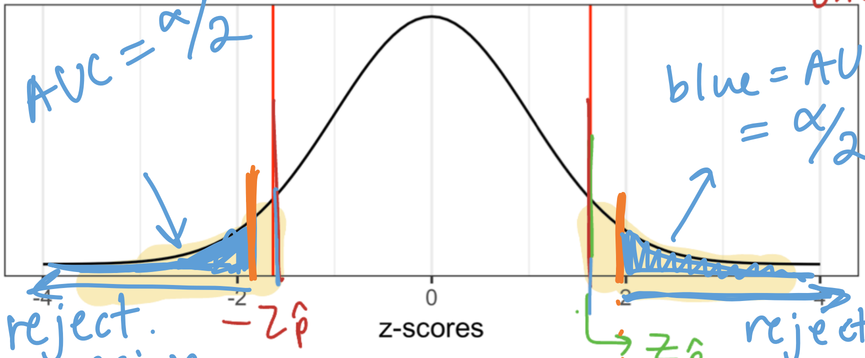
0.105



AUC = $\alpha/2$

blue = AUC = $\alpha/2$

reject region (z-values lower than z^*)

$-z_{\hat{p}}$

$z_{\hat{p}}$

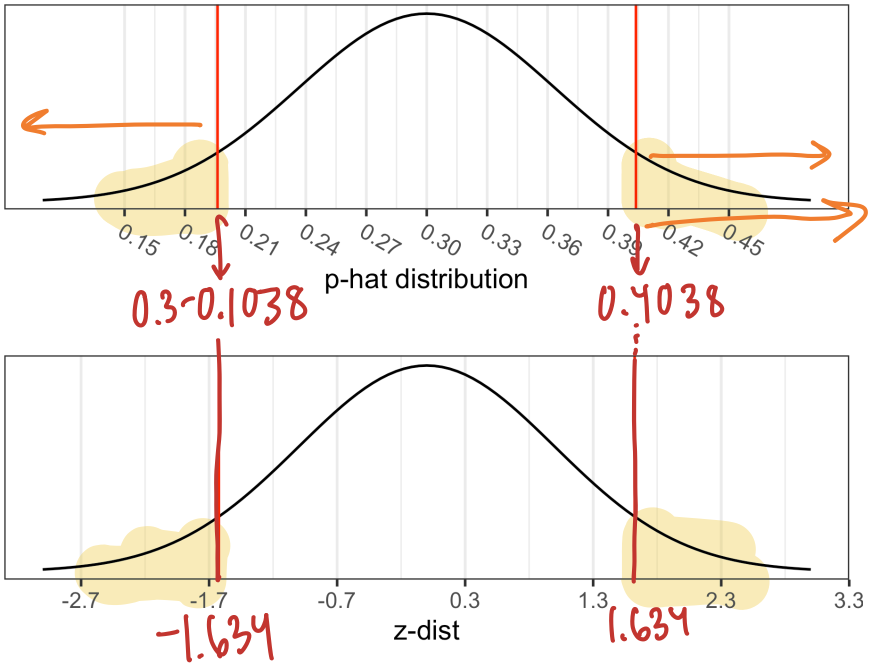
rejection region (z vals greater than z^*)

1.634

$z^* = 1.96$

Step 5: p-value

The **p-value** is the **probability** of obtaining a test statistic *just as extreme or more extreme* than the observed test statistic assuming the null hypothesis H_0 is true.



Calculate the *p*-value:

$$
\begin{aligned}
&2 \cdot P(\hat{p} > 0.404) \\
&= 2 \cdot P\left(Z_{\hat{p}} > \frac{0.404 - 0.30}{\sqrt{\frac{0.30 \cdot (1 - 0.30)}{52}}}\right) \\
&= 2 \cdot P(Z_{\hat{p}} > 1.634) \\
&= 0.1022348
\end{aligned}
$$

```
1  2*pnorm(1.634, lower.tail = F)
```

```
[1] 0.1022589
```

The prob of getting sample proportion more extreme than 0.4038 is 0.1023.

Step 4-5: test statistic and p-value together using `prop.test()`

- This will match our "by-hand" calculation
- So far we have not used the continuity correction, but this will give us a different answer
- **Conitinuity correction is more widely accepted, so I suggest using R to calculate the test statistic and p-value when you can!**

```
prop.test(x = 21, n = 52, p = 0.30, correct = F)
```

No CC.

```
	1-sample proportions test without continuity correction

data:  21 out of 52, null probability 0.3
X-squared = 2.6703, df = 1, p-value = 0.1022
alternative hypothesis: true p is not equal to 0.3
95 percent confidence interval:
 0.2815973 0.5393242
sample estimates:
        p
0.4038462
```

DO NOT USE THIS SLIDE AS REF.

Step 4-5: test statistic and p-value together using `prop.test()`

USING THE CONTINUITY CORRECTION

$H_0: p = 0.30$

```
prop.test(x = 21, n = 52, p = 0.30, correct = T)
```

p_0

```
	1-sample proportions test with continuity correction

data:  21 out of 52, null probability 0.3
X-squared = 2.1987, df = 1, p-value = 0.1381
alternative hypothesis: true p is not equal to 0.3
95 percent confidence interval:
 0.2731269 0.5487141
sample estimates:
        p 
0.4038462 
```

▶ Tidying the output of `prop.test()`

estimate	statistic	p.value	parameter	conf.low	conf.high	method	alternative
0.4038462	2.198718	0.1381256	1	0.2731269	0.5487141	1-sample proportions test with continuity correction	two.sided

Step 6: Conclusion to hypothesis test

$$H_0: p = 0.30$$
$$H_A: p \neq 0.30$$

- Recall the p-value = 0.1022348
- Use α = 0.05.
- Do we reject or fail to reject H_0?

0.1381 w/ CC

p-val is 0.1381 $> 0.05 = \alpha$

Fail to Reject

Conclusion statement:

- Stats class conclusion
  - There is insufficient evidence that the (population) proportion of individuals who had an immune response is different than 0.30 (p-value = 0.102).
- More realistic manuscript conclusion:
  - In a sample of 52 individuals receiving treatment, 40.4% had an immune response, which is not different from 30% (p-value = 0.102).

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Conditions for one proportion: test vs. CI

Confidence interval conditions

1. *Independent observations*
   - The observations were collected independently.

2. The number of successes and failures is at least 10:

$$n\hat{p} \geq 10, \quad n(1-\hat{p}) \geq 10$$

Hypothesis test conditions

1. *Independent observations*
   - The observations were collected independently.

2. The number of **expected** successes and **expected** failures is at least 10.

$$np_0 \geq 10, \quad n(1-p_0) \geq 10$$

b/c we use Norm approx under the null

95% CI for population proportion

What to use for SE in CI formula?

$$\hat{p} \pm z^* \cdot SE_{\hat{p}}$$

Sampling distribution of $\hat{p}$:

we do not know p,
then use sample sd/SE

$$\hat{p} \sim N\left(\mu_{\hat{p}} = p, \sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}}\right)$$

Problem: We don't know what p is - it's what we're estimating with the CI.
Solution: approximate p with $\hat{p}$:

$$SE_{\hat{p}} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- Note that I am not using a continuity correction here! This means our "by hand" calculation will be different than our R calculation
  - Using the continuity correction is more widely accepted
  - So I would suggest using R to calculate the confidence intervals when you can!

95% CI for population proportion of immune response by hand

95% CI for population proportion p:

$$\hat{p} \pm z^* \cdot SE_{\hat{p}}$$

$$\hat{p} \pm z^* \cdot \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$0.404 \pm 1.96 \cdot \sqrt{\frac{0.404(1-0.404)}{52}}$$

$$0.404 \pm 1.96 \cdot 0.068$$

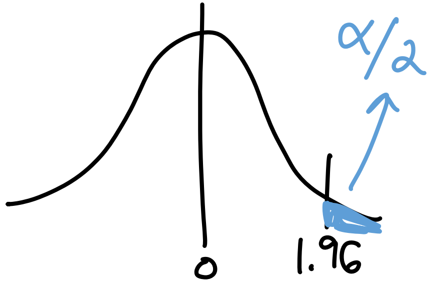
$$0.404 \pm 0.133$$

$$(0.27, 0.537)$$

$\hat{p} = 0.4038$

$n = 52$

Used z^* = `qnorm(0.975)` = 1.96



"By hand" Conclusion:
We are 95% confident that the (population) proportion of individuals with an immune response is between 0.27 and 0.537.

95% CI for population proportion of immune response using R

- We can use R to get similar values

no p inputted

```
1 prop.test(x = 21, n = 52, conf.level = 0.95, correct = T)
```

default

```
	1-sample proportions test with continuity correction

data:  21 out of 52, null probability 0.5
X-squared = 1.5577, df = 1, p-value = 0.212
alternative hypothesis: true p is not equal to 0.5
95 percent confidence interval:
 0.2731269 0.5487141
sample estimates:
        p
0.4038462
```

R Conclusion:
We are 95% confident that the (population) proportion of individuals with an immune response is between 0.273 and 0.549.

- Note: We expect some differences between the confidence interval calculated by hand vs. by R. R uses a slightly different method to calculate.

Break Time!

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Inference for difference of two independent proportions

$$\hat{p}_1 - \hat{p}_2$$

- For means, we went from *inferences on single sample mean* to *inferences on difference in means from two independent samples*
- We can do the same thing for proportions
- We will go from *inferences on single sample proportion* to *inferences on difference in proportions from two independent samples*

Poll Everywhere Question 3

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For inference for means, we had single sample mean, paired mean difference, and difference in means. Why don't we have a paired proportion difference?

Paired mean difference takes the difference of individual samples first before taking the sample mean. Proportions are measures of the whole sample already

Proportions are comparing two different individuals?

proportion is a value taken from the sample as a whole. so there isn't really a such thing as a proportion taken from an individual.

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For difference in proportions: Population parameters vs. sample statistics

Population parameter

- Population 1 proportion: p_1, π_1 ("pi")
- Population 2 proportion: p_2, π_2 ("pi")

- Difference in proportions: $p_1 - p_2$

Sample statistic (point estimate)

- Sample 1 proportion: $\hat{p}_1, \hat{\pi}_1$ ("pi")
- Sample 1 proportion: $\hat{p}_2, \hat{\pi}_2$ ("pi")

- Difference in proportions: $\hat{p}_1 - \hat{p}_2$

Sampling distribution of $\hat{p}_1 - \hat{p}_2$

- $\hat{p}_1 = \frac{X_1}{n_1}$ and $\hat{p}_2 = \frac{X_2}{n_2}$,
  - X_1 & X_2 are the number of "successes"
  - n_1 & n_2 are the sample sizes of the 1st & 2nd samples

- Each $\hat{p}$ can be approximated by a normal distribution, for "big enough" n
- Since the difference of independent normal random variables is also normal, it follows that for "big enough" n_1 and n_2

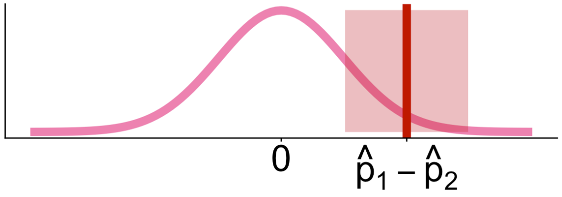
$$\hat{p}_1 - \hat{p}_2 \sim N\left(\mu_{\hat{p}_1-\hat{p}_2} = p_1 - p_2,\ \sigma_{\hat{p}_1-\hat{p}_2} = \sqrt{\frac{p_1 \cdot (1-p_1)}{n_1} + \frac{p_2 \cdot (1-p_2)}{n_2}}\right)$$

- What is "big enough"? At least 10 successes and 10 failures are expected in the sample: $n_1 p \geq 10$, $n_1(1-p) \geq 10$, $n_2 p \geq 10$, and $n_2(1-p) \geq 10$

Approaches to answer a research question

- **Research question is a generic form for a single proportion:** Is there evidence to support that the population proportions are different from each other?

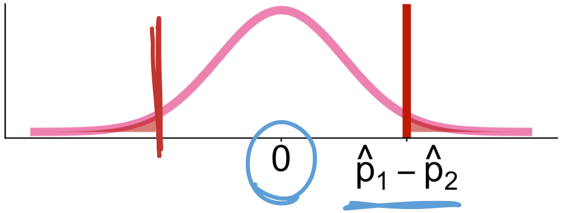
Calculate **CI for the proportion difference** $p_1 - p_2$:



$$\hat{p}_1 - \hat{p}_2 \pm z^* \cdot SE_{\hat{p}_1 - \hat{p}_2}$$

- with z^* = z-score that aligns with specific confidence interval

Run a **hypothesis test:**



Hypotheses

$$H_0 : p_1 - p_2 = 0$$
$$H_A : p_1 - p_2 \neq 0$$
$$(or \; <, >)$$

Test statistic

$$z_{\hat{p}_1 - \hat{p}_2} = \frac{\hat{p}_1 - \hat{p}_2}{SE_{pool}}$$

Learning Objectives

1. Remind ourselves of the Normal approximation of the binomial distribution and define the sampling distribution of a sample proportion
2. Run a hypothesis test for a single proportion and interpret the results.
3. Construct and interpret confidence intervals for a single proportion.
4. Understand how CLT applies to a difference in binomial random variables
5. Run a hypothesis test for a difference in proportions and interpret the results.
6. Construct and interpret confidence intervals for a difference in proportions.

Motivating example: effectiveness of mammograms

A 30-year study to investigate the effectiveness of mammograms versus a standard non-mammogram breast cancer exam was conducted in Canada with 89,835 participants. Each person was randomized to receive either annual mammograms or standard physical exams for breast cancer over a 5-year screening period.

By the end of the 25-year follow-up period, 1,005 people died from breast cancer. The results are summarized in the following table.

▶ Displaying the contingency table in R

	Death from breast cancer?		
Group	Yes	No	Total
Control Group	505	44405	44910
Mammogram Group	500	44425	44925
Total	1005	88830	89835

Reference: Steps in a Hypothesis Test

1. Check the **assumptions**
2. Set the **level of significance** α
3. Specify the **null** (H_0) and **alternative** (H_A) **hypotheses**
    1. In symbols
    2. In words
    3. Alternative: one- or two-sided?
4. Calculate the **test statistic**.
5. Calculate the **p-value** based on the observed test statistic and its sampling distribution
6. Write a **conclusion** to the hypothesis test
    1. Do we reject or fail to reject H_0?
    2. Write a conclusion in the context of the problem

Before we start, we need to calculate the pooled proportion

- Often, our null hypothesis is that the two proportions are equal
  - And that both populations are the same
- Thus, we calculate a pooled proportion to represent the proportion under the null distribution

$$\text{pooled proportion} = \hat{p}_{pool} = \frac{\text{total number of ``successes''}}{\text{total number of cases}} = \frac{x_1 + x_2}{n_1 + n_2}$$

- In this example:

$$\hat{p}_{pool} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{500 + 505}{(500 + 44425) + (505 + 44405)} = 0.01119$$

Poll Everywhere Question 4

14:46 Mon Nov 24 12%

Join by Web **PollEv.com/nickywakim275**

Why do we use the pooled proportion when we run a hypothesis test?

Under the null hypothesis, the two proportions are equal ✓	88%
Under the null hypothesis, the two proportions are not equal	0%
Under the alternative hypothesis, the two proportions are equal	0%
Under the alternative hypothesis, the two proportions are not equal	12%

Powered by Poll Everywhere

means

$H_0: \mu_1 = \mu_2$

props

$H_0: p_1 = p_2$

Step 1: Check the assumptions

Conditions:

- *Independent observations & samples*
  - The observations were collected independently.
  - In particular, observations from the two groups weren't paired in any meaningful way.
- The number of expected successes and expected failures is at least 10 *for each group* - using the pooled proportion:
  - $n_1\hat{p}_{pool} \geq 10, \quad n_1(1-\hat{p}_{pool}) \geq 10$
  - $n_2\hat{p}_{pool} \geq 10, \quad n_2(1-\hat{p}_{pool}) \geq 10$

- In the example, we check:
  - $n_1\hat{p}_{pool} = 44925 \cdot 0.0112 = 502.5839 \geq 10$
  - $n_1(1-\hat{p}_{pool}) = 44925(1-0.0112) = 44422.42 \geq 10$
  - $n_2\hat{p}_{pool} = 44910 \cdot 0.0112 = 502.4161 \geq 10$
  - $n_2(1-\hat{p}_{pool}) = 44910(1-0.0112) = 44407.58 \geq 10$

Step 3: Null and Alternative Hypothesis test

Two samples test

- H_0: The difference in population proportions of deaths from breast cancer among people who received annual mammograms and annual physical check-ups is 0.
- H_A: The difference in population proportions of deaths from breast cancer among people who received annual mammograms and annual physical check-ups is not 0.

$$H_0 : p_{mamm} - p_{ctrl} = 0$$
$$H_A : p_{mamm} - p_{ctrl} \neq 0$$

$p_{mamm} = p_{ctrl}$

$p_{mamm} \neq p_{ctrl}$

Step 4: Test statistic (1/2)

Sampling distribution of $\hat{p}_1 - \hat{p}_2$:

$$\hat{p}_1 - \hat{p}_2 \sim N\left(\mu_{\hat{p}_1 - \hat{p}_2} = p_1 - p_2, \ \sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{p_1 \cdot (1 - p_1)}{n_1} + \frac{p_2 \cdot (1 - p_2)}{n_2}}\right)$$

Since we assume $H_0 : p_1 - p_2 = 0$ is true, we "pool" the proportions of the two samples to calculate the SE:

$$\text{pooled proportion} = \hat{p}_{pool} = \frac{\text{total number of successes}}{\text{total number of cases}} = \frac{x_1 + x_2}{n_1 + n_2}$$

Test statistic:

$$\text{test statistic} = z_{\hat{p}_1 - \hat{p}_2} = \frac{\hat{p}_1 - \hat{p}_2 - 0}{\sqrt{\frac{\hat{p}_{pool}(1 - \hat{p}_{pool})}{n_1} + \frac{\hat{p}_{pool}(1 - \hat{p}_{pool})}{n_2}}}$$

under null, SE uses $\hat{p}_{pool}$

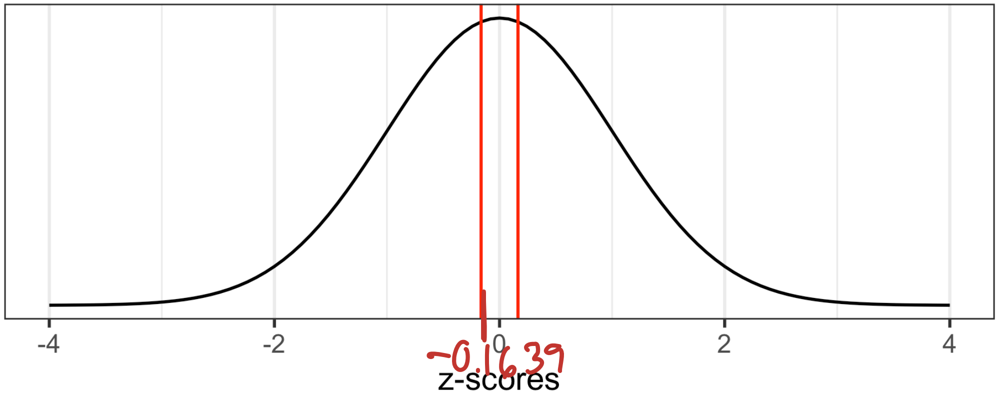
Step 4: Test statistic (2/2)

From our example: Recall that $\hat{p}_1 = \frac{500}{44925} = 0.0111, \hat{p}_2 = \frac{505}{44910} = 0.0112, n_1 = 44925, n_2 = 44910$, and $\hat{p}_{pool} = 0.01119$

The test statistic is:

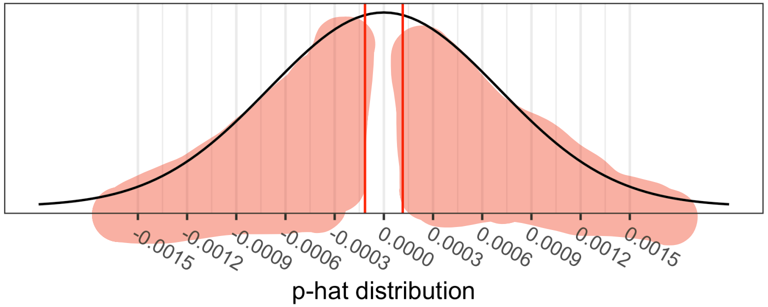
$$z_{\hat{p}_1-\hat{p}_2} = \frac{\hat{p}_1 - \hat{p}_2 - 0}{\sqrt{\frac{\hat{p}_{pool}\cdot(1-\hat{p}_{pool})}{n_1} + \frac{\hat{p}_{pool}\cdot(1-\hat{p}_{pool})}{n_2}}} = \frac{0.0111 - 0.0112}{\sqrt{\frac{0.01119\cdot(1-0.01119)}{44925} + \frac{0.01119\cdot(1-0.01119)}{44910}}} = -0.163933$$

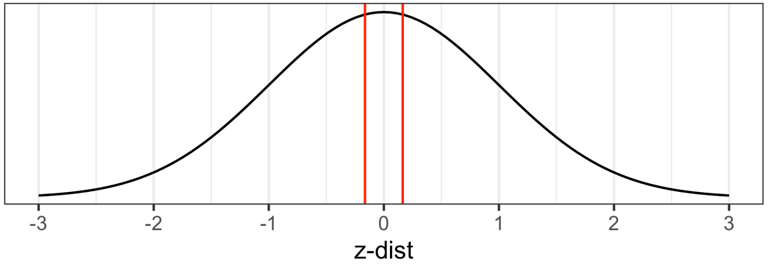
- Let's see the z-score on a Z-distribution (Standard Normal curve)



Step 5: p-value

The **p-value** is the **probability** of obtaining a test statistic *just as extreme or more extreme* than the observed test statistic assuming the null hypothesis H_0 is true.





Calculate the *p*-value:

$$2 \cdot P(\hat{p}_1 - \hat{p}_2 < 0.0111 - 0.0112)$$
$$= P\left(Z_{\hat{p}_1 - \hat{p}_2} < \frac{0.0111 - 0.0112}{\sqrt{\frac{0.01119\cdot(1-0.01119)}{44925} + \frac{0.01119\cdot(1-0.01119)}{44910}}}\right)$$
$$= 2 \cdot P(Z_{\hat{p}} > -0.164)$$
$$= 0.8697839$$

```
1 2*pnorm(-0.1639)
```

```
[1] 0.8698099
```

Step 4-5: test statistic and p-value together using `prop.test()`

```
prop.test(x = c(505, 500), n = c(44910, 44925)) # no p needed
```

x_1 x_2 n_1 n_2

```
	2-sample test for equality of proportions with continuity correction

data:  c(505, 500) out of c(44910, 44925)
X-squared = 0.01748, df = 1, p-value = 0.8948
alternative hypothesis: two.sided
95 percent confidence interval:
 -0.001282751  0.001512853
sample estimates:
    prop 1     prop 2 
0.01124471 0.01112966 
```

▶ Tidying the output of `prop.test()`

estimate1	estimate2	statistic	p.value	parameter	conf.low	conf.high	method	alternative
0.01124471	0.01112966	0.01747975	0.8948174	1	-0.001282751	0.001512853	2-sample test for equality of proportions with continuity correction	two.sided

- Note: We expect some differences between the test statistic and p-value calculated by hand vs. by R bc R uses the continuity correction

Step 6: Conclusion to hypothesis test

$$H_0 : p_{mamm} - p_{ctrl} = 0$$
$$H_A : p_{mamm} - p_{ctrl} \neq 0$$

- Recall the p-value = 0.8698
- Use α = 0.05
- Do we reject or fail to reject H_0?

Conclusion statement:

- Stats class conclusion
  - There is insufficient evidence that the difference in (population) proportions of deaths from breast cancer among people who received annual mammograms and annual physical check-ups different (p-value = 0.87).
- More realistic manuscript conclusion:
  - 1.11% of people receiving annual mammograms (n=44925) and 1.12% of people receiving annual physical exams (n=44925) died from breast cancer (p-value = 0.87).

Learning Objectives

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4. Understand how CLT applies to a difference in binomial random variables
5. Run a hypothesis test for a difference in proportions and interpret the results.
6. Construct and interpret confidence intervals for a difference in proportions.

Conditions for difference in proportions: test vs. CI

Confidence interval conditions

1. *Independent observations & samples*
   - The observations were collected independently.
   - In particular, observations from the two groups weren't paired in any meaningful way.
2. The number of successes and failures is at least 10 *for each group.*
   - $n_1\hat{p}_1 \geq 10, \;\; n_1(1-\hat{p}_1) \geq 10$
   - $n_2\hat{p}_2 \geq 10, \;\; n_2(1-\hat{p}_2) \geq 10$

Hypothesis test conditions

1. *Independent observations & samples*
   - The observations were collected independently.
   - In particular, observations from the two groups weren't paired in any meaningful way.
2. The number of **expected** successes and **expected** failures is at least 10 *for each group* - using the pooled proportion:
   - $n_1\hat{p}_{pool} \geq 10, \;\; n_1(1-\hat{p}_{pool}) \geq 10$
   - $n_2\hat{p}_{pool} \geq 10, \;\; n_2(1-\hat{p}_{pool}) \geq 10$

Poll Everywhere Question 5

95% CI for population difference in proportions

What to use for SE in CI formula?

$$\hat{p}_1 - \hat{p}_2 \pm z^* \cdot SE_{\hat{p}_1 - \hat{p}_2}$$

SE in sampling distribution of $\hat{p}_1 - \hat{p}_2$

$$\sigma_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{p_1 \cdot (1 - p_1)}{n_1} + \frac{p_2 \cdot (1 - p_2)}{n_2}}$$

Problem: We don't know what p is - it's what we're estimating with the CI.
Solution: approximate p_1, p_2 with $\hat{p}_1, \hat{p}_2$:

$$SE_{\hat{p}_1 - \hat{p}_2} = \sqrt{\frac{\hat{p}_1 \cdot (1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2 \cdot (1 - \hat{p}_2)}{n_2}}$$

95% CI for the population difference in proportions

95% CI for population mean difference $p_1 - p_2$:

$$
\begin{gathered}
\hat{p}_1 - \hat{p}_2 \pm z^* \cdot SE_{\hat{p}_1 - \hat{p}_2} \\
\hat{p}_1 - \hat{p}_2 \pm z^* \cdot \sqrt{\frac{\hat{p}_1 \cdot (1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2 \cdot (1 - \hat{p}_2)}{n_2}} \\
0.01113 - 0.01124 \pm 1.96 \cdot \sqrt{\frac{0.01113 \cdot (1 - 0.01113)}{44925} + \frac{0.01124 \cdot (1 - 0.01124)}{44910}} \\
0.35 \pm 1.96 \cdot 0.001 \\
0.35 \pm 0.002 \\
(-0.002, 0.002)
\end{gathered}
$$

Used z^* = `qnorm(0.975)` = 1.96

Interpretation:
We are 95% confident that the difference in (population) proportions of deaths due to breast cancer comparing people who received annual mammograms to annual physical check-ups is between -0.002 and 0.002.

95% CI for the population difference in proportions

- We can use R to get similar values

```
prop.test(x = c(505, 500), n = c(44910, 44925))
```

```

	2-sample test for equality of proportions with continuity correction

data:  c(505, 500) out of c(44910, 44925)
X-squared = 0.01748, df = 1, p-value = 0.8948
alternative hypothesis: two.sided
95 percent confidence interval:
 -0.001282751  0.001512853
sample estimates:
    prop 1     prop 2 
0.01124471 0.01112966 
```

R Conclusion:
We are 95% confident that the difference in (population) proportions of deaths due to breast cancer comparing people who received annual mammograms to annual physical check-ups is between -0.0013 and 0.0015.

- Note: We expect some differences between the confidence interval calculated by hand vs. by R. R uses a slightly different method to calculate.