Chapter 1: Outcomes, Events, and Sample Spaces

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Class Overview

- Tossing One Coin (Outcomes, Events, and Sample Space)
- Tossing Two Coins (Outcomes, Events, and Sample Space)
- Set Theory

definition

Tossing One Coin (Outcomes, Events, and Sample Space)

Coin Toss Example: 1 coin (1/3)

Suppose you toss one coin.

- What are the possible outcomes?

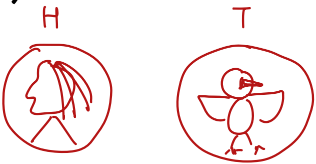
- What is the sample space?

- What are the possible events?

Coin Toss Example: 1 coin (2/3)

Suppose you toss one coin.

- What are the possible outcomes?
  - Heads (H)
  - Tails (T)



> **Note**
>
> When something happens at random, such as a coin toss, there are several possible outcomes, and *exactly one* of the outcomes will occur.

occuring $\neq$ event

not necessarily

Coin Toss Example: 1 coin (3/3)

H T

Definition: Sample Space

The **sample space** S is the set of *all* outcomes

Definition: Event

An **event** is a *collection of some* outcomes. An event can include multiple outcomes or no outcomes.

- What is the sample space?
  - $S = \{H, T\}$

→ denotes a set: collecting a list of outcomes or values

- What are the possible events?
  - H → {H}
  - T {T}
  - {H, T}
  - $\emptyset$ empty set (neither heads nor tails) (nothing from S)

When thinking about events, think about outcomes that you might be asking the probability of.

$2^{|S|}$ → total # of outcomes in sample space

→ $|S| = 2$ $2^2 = 4$

6 sided die

$S = \{1, 2, 3, 4, 5, 6\}$

$2^{|S|} = 2^6$

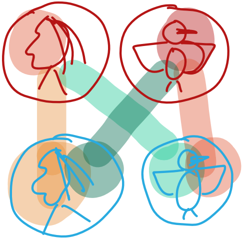
Tossing Two Coins (Outcomes, Events, and Sample Space)

Coin Toss Example: 2 coins

Suppose you toss two coins.

coin 1:

coin 2:



- What is the sample space? *Assume the coins are distinguishable*
  - $S = \{HH, HT, TH, TT\}$

- What are some possible events?
  - A = exactly one H = $\{HT, TH\}$
  - B = at least one H = $\{HH, HT, TH\}$
  -
  - C = no heads = $\{TT\}$

if NOT distinguishable:

$S = \{HH, HT, TT\}$

HH HT TH TT

$\to 2\times 2\times 2\times 2 = 2^4$

$\{HH\}, \{HT\}, \{TH\}, \{TT\}$

$2^{|S|} = 2^4 = 16 \to \{HH, HT\}, \{HH, TH\}, \{HH, TT\}$

More info on events and sample spaces

- We usually use capital letters from the beginning of the alphabet to denote events. However, other letters might be chosen to be more descriptive.

- We use the notation $|S|$ to denote the **size** of the sample space.

$|S|$ $|A|$

- The total number of possible events is $2^{|S|}$, which is the total number of possible subsets of S. We will prove this later in the course.

- The **empty set**, denoted by $\emptyset$, is the set containing no outcomes.

$\emptyset$

Example: Keep sampling until...

Suppose you keep sampling people until you have someone with high blood pressure (BP)

What is the sample space?

- Let H = denote someone with high BP.
- Let H^{C} = denote someone with not high blood pressure, such as low or regular BP.

  H' $\quad S \setminus H$

- Then, S = $\{H, H^cH, H^cH^cH, H^cH^cH^cH, \ldots\}$

$|S| = \infty$

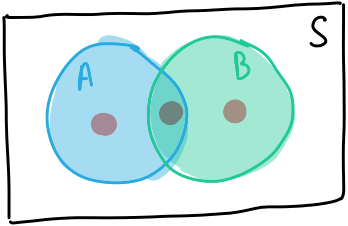
Set Theory

Set Theory (1/2)

Venn diagrams

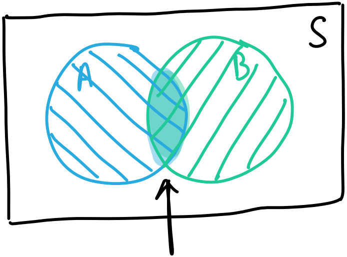
Definition: Union

The **union** of events A and B, denoted by $\mathrm{A} \cup \mathrm{B}$, contains all outcomes that are in A *or* B or both



Definition: Intersection

The **intersection** of events A and B, denoted by $\mathrm{A} \cap \mathrm{B}$, contains all outcomes that are both in A *and* B.

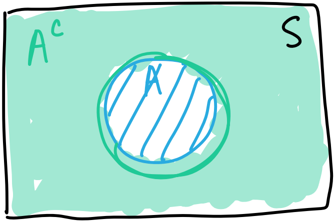


Set Theory (2/2)

Venn diagrams

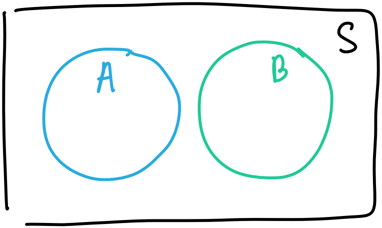
Definition: Complement

The **complement** of event A, denoted by A^{C} or A', contains all outcomes in the sample space S that are *not* in A .



Definition: Mutually Exclusive

Events A and B are **mutually exclusive**, or disjoint, if they have no outcomes in common. In this case $\mathrm{A} \cap \mathrm{B} = \emptyset$, where $\emptyset$ is the empty set.



BP example variation (1/3)

- Suppose you have n subjects in a study.
- Let $\mathrm{H_i}$ be the event that person i has high BP, for $\mathrm{i = 1 \ldots n}$.

Use set theory notation to denote the following events:

1. Event subject i does not have high BP
2. Event all n subjects have high BP
3. Event at least one subject has high BP
4. Event all of them do not have high BP
5. Event at least one subject does not have high BP

BP example variation (2/3)

- Suppose you have n subjects in a study.
- Let H_i be the event that person i has high BP, for $i = 1 \ldots n$.

Use set theory notation to denote the following events:

1. Event subject i does not have high BP

H_i^c or H_i' = no high BP for person i

2. Event all n subjects have high BP

$$H_1 \cap H_2 \cap H_3 \cap \cdots \cap H_n = \bigcap_{i=1}^{n} H_i$$

person 1 & 2

$H_1 \cap H_2$

3. Event at least one subject has high BP

$$H_1 \cup H_2 \cup H_3 \cup \cdots \cup H_n = \bigcup_{i=1}^{n} H_i$$

$$\sum_{i=1}^{n} X_i = X_1 + X_2 + X_3 + \cdots + X_n$$

person 1 & 2

$H_1 \cup H_2$

$H_1 H_2^c$ $H_1 H_2$

$H_1^c H_2$

BP example variation (3/3)

$\bigcap$ all do have H

$\bigcap$ all do not H^c

4. Event all of them do not have high BP

$$H_1^c \cap H_2^c \cap H_3^c \cap \dots \cap H_n^c = \bigcap_{i=1}^{n} H_i^c = \left(\bigcup_{i=1}^{n} H_i\right)^c$$

p 1 & 2

$H_1 \quad H_1^c$

$H_2 \quad H_2^c$

$H_1^c \cap H_2^c \rightarrow H_1^c H_2^c$

$H_1 \cup H_2 \rightarrow H_1H_2,\ H_1^cH_2,\ H_1H_2^c$

c

5. Event at least one subject does not have high BP

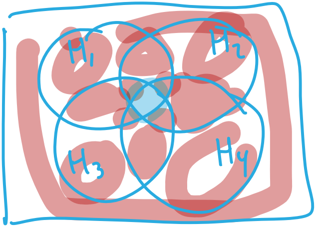
P1 & 2 : $H_1H_2^c \quad H_1^cH_2 \quad H_1^cH_2^c$

complement: H_1H_2

$$H_1^c \cup H_2^c \cup H_3^c \cup \dots \cup H_n^c = \bigcup_{i=1}^{n} H_i^c$$

$\downarrow$

$$\left(\bigcap_{i=1}^{n} H_i\right)^c$$



De Morgan's Laws

Theorem: De Morgan's 1st Law

For a collection of events (sets) $A_1, A_2, A_3, \ldots$

$$\bigcap_{i=1}^{n \,\infty} A_i^C = \left(\bigcup_{i=1}^{n} A_i \right)^C$$

intersection of complements is comp. of union

"all not A = (at least one event A)^C"

Theorem: De Morgan's 2nd Law

For a collection of events (sets) $A_1, A_2, A_3, \ldots$

$$\bigcup_{i=1}^{n \,\infty} A_i^C = \left(\bigcap_{i=1}^{n \,\infty} A_i \right)^C$$

6 sided

$A = \{2, 4, 6\}$

$A^c = \{1, 3, 5\}$

"at least one event not A = (all A)^C"

Remarks on De Morgan's Laws

- These laws also hold for infinite collections of events.

- Draw Venn diagrams to convince yourself that these are true!

- These laws are *very* useful when calculating probabilities.
  - This is because calculating the probability of the **intersection of events is often much easier than the union of events**.
  - This is not obvious right now, but we will see in the coming chapters why.