Chapter 2: Probability

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2023-09-27

Class Overview

- Probabilities of equally likely events
- Probability Axioms
- Some probability properties
- Partitions
- Venn Diagram Probabilities

Probabilities of equally likely events

Pick an *equally likely* card, any *equally likely* card

Example 1

Suppose you have a regular well-shuffled deck of cards. What's the probability of drawing:

1. any heart
2. the queen of hearts
3. any queen

sample space: 52 cards

4 suits: ♡ ◇ ♣ ♠

13 faces/#s: A 2-10 J Q K

① $P(♡) = \frac{13}{52} = \frac{1}{4} = 0.25$

probability of ( )

② $P(Q\,♡) = \frac{1}{52}$

③ $P(Q) = \frac{4}{52} = \frac{1}{13}$

Let's break down this probability

If S is a finite sample space, with **equally likely outcomes**, then

$$\mathbb{P}(A) = \frac{|A|}{|S|}. \qquad \frac{\text{total \# of outcomes in } A}{\text{total \# of outcome in } S}$$

$P \quad \mathbb{P} \quad \mathbf{P} \quad prob()$

A probability is a function...

A contained w/in S
A is a subset of S

$\mathbb{P}(A)$ is a function with

- **Input:** event A from the sample space S, $(A \subseteq S)$
- **Output:** a number between 0 and 1 (inclusive)

$$\mathbb{P}(A) : S \to [0, 1]$$

A function that follows some specific rules though!

See Probability Axioms on next slide.

Probability Axioms

Probability Axioms

Axiom 1

For every event A, $0 \le \mathbb{P}(A) \le 1$.

Axiom 2

$P(S) = \frac{|S|}{|S|} = 1$

For the sample space S, $\mathbb{P}(S) = 1$.

Axiom 3

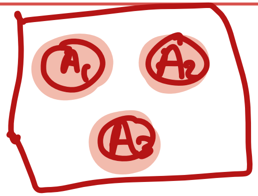
If $A_1, A_2, A_3, \ldots,$ is a collection of **disjoint** events, then

$P(A_1 \cup A_2 \cup A_3)$

$= P(A_1) + P(A_2) + P(A_3)$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

probability of at least one
$A_1 - A_n$ happening



Some probability properties

Some probability properties

Using the Axioms, we can prove all other probability properties!

Proposition 1

For any event A, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$

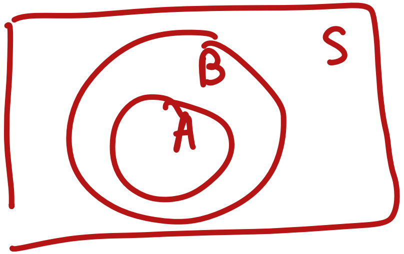
Proposition 2

$\mathbb{P}(\emptyset) = 0$

Proposition 3

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$

$\subset$ $\subseteq$



Proposition 4

A & B not necessarily disjoint

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

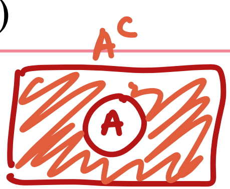
Proposition 5

$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$

Proposition 1 Proof

Proposition 1

For any event A, $\mathbb{P}(A) = 1 - \mathbb{P}(A^C)$



$\underline{A \cup A^c = S}$

A & A^c are disjoint

$P(\underline{A \cup A^c}) = P(A) + P(A^c)$

$P(S) = P(A) + P(A^c)$

$1 = P(A) + P(A^c)$

$-P(A^c) \qquad -P(A^c)$

$P(A) = 1 - P(A^c)$

AXIOMS

A1: $0 \le P(A) \le 1$

A2: $P(S) = 1$

A3: $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

for disjoint A_i's

Proposition 2 Proof

Proposition 2

$\mathbb{P}(\emptyset) = 0$

prop 1: $P(A) = 1 - P(A^c)$

$A = \underline{\emptyset} \quad A^c = \underline{S}$

$P(\emptyset) = 1 - P(S)$

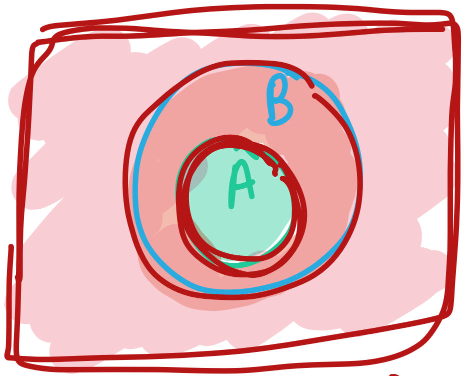
$= 1 - 1$

$P(\emptyset) = 0$

Proposition 3 Proof

Proposition 3

If $A \subseteq B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$



$\rightarrow B = A \cup (A^c \cap B)$ green + orange

$+$

disjoint events

$$P(B) = P(A \cup (A^c \cap B)) = P(A) + P(A^c \cap B)$$

$$P(B) = P(A) + \underbrace{P(A^c \cap B)}$$

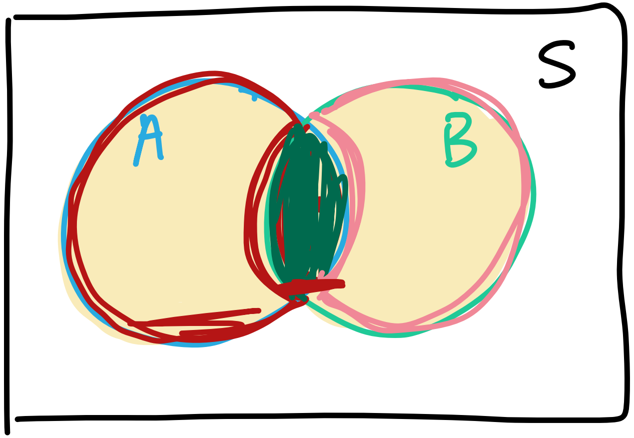
AX 1: $0 \leq P(A^c \cap B) \leq 1$

$$P(A) = P(B) - \underset{\leq 0}{P(A^c \cap B)} \leq 0 \Rightarrow P(A) \leq P(B)$$

Proposition 4 Visual Proof

Proposition 4 disjoint events or NOT disjoint

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$



$A \cap B^c$

$A \cap B$

$B \cap A^c$

$P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(B \cap A^c)$

$P(A) + P(B) - P(A \cap B)$

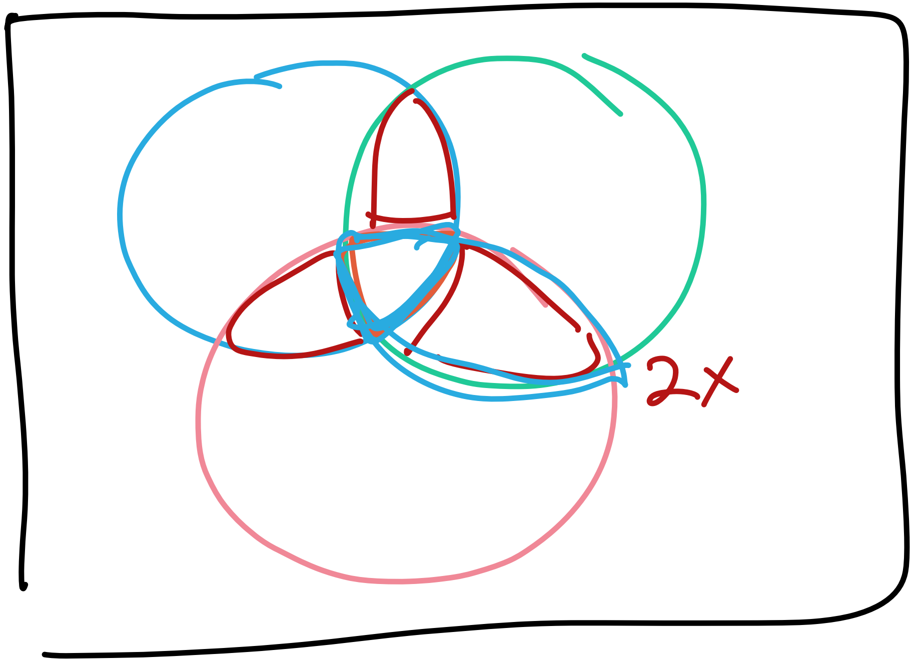
$P(A \cap B^c) + P(A \cap B)$

$P(B \cap A^c) + P(A \cap B)$

Proposition 5 Visual Proof

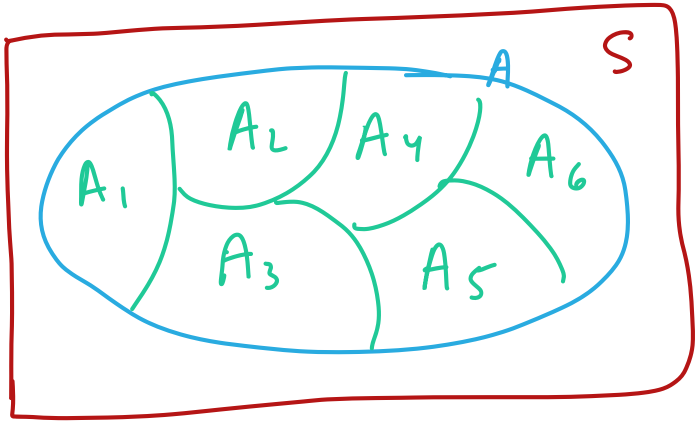
Proposition 5

$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) + \mathbb{P}(A \cap B \cap C)$



Partitions

Partitions



Definition: Partition

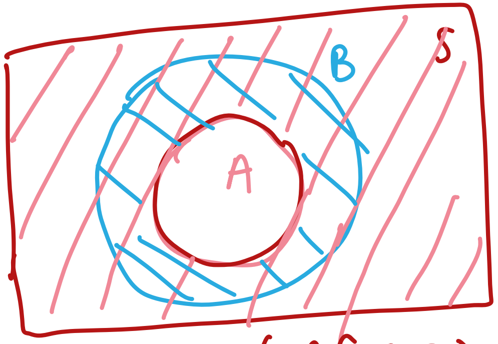
A set of events $\{A_i\}_{i=1}^{n}$ create a **partition** of A, if

- the A_i's are disjoint (mutually exclusive) and
- $\boxed{\bigcup_{i=1}^{n} A_i} = A$

Example 2

the set $\{O, O\}$

- If $A \subset B$, then $\{A, B \cap A^C\}$ is a partition of B.
- If $S = \bigcup_{i=1}^{n} A_i$, and the A_i's are disjoint, then the A_i's are a partition of the sample space.



$A \cup (A^c \cap B)$

Creating partitions is sometimes used to help calculate probabilities, since by Axiom 3 we can add the probabilities of disjoint events.

Venn Diagram Probabilities

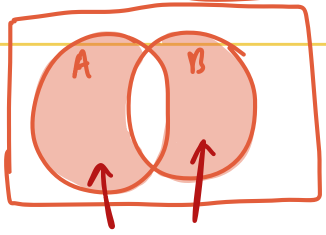
Weekly medications

Example 3

If a subject has an

- 80% chance of taking their medication *this* week,
- 70% chance of taking their medication *next* week, and
- 10% chance of *not* taking their medication *either* week,

then find the probability of them taking their medication exactly one of the two weeks.



Hint: Draw a Venn diagram labelling each of the parts to find the probability.

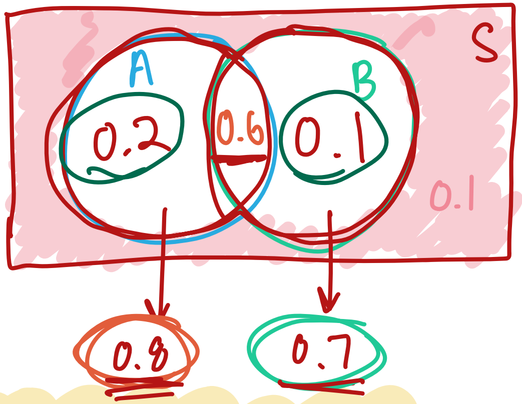
Let A = take medication this week

B = take med. next week

$A^c \cap B^c \longrightarrow P(A^c \cap B^c) = 0.10$

dont take this week

dont take next week



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$1 - P(A^c \cap B^c)$

$1 - 0.1 = 0.9$

$0.9 = 0.8 + 0.7 - P(A \cap B)$

$-0.9 + P(A \cap B)$ $\quad$ $+P(A \cap B) - 0.9$

$P(A \cap B) = 1.5 - 0.9 = 0.6$

P(exactly one of two weeks) = 0.3

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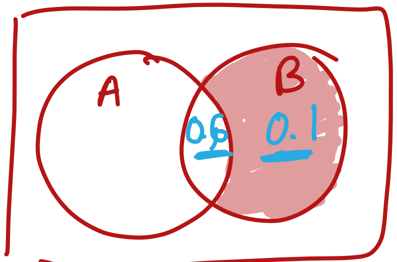
then find the probability of them taking their medication exactly one of the two weeks.

① 1st week take (but not 2nd)

② 2nd week take (but not 1st) → ②

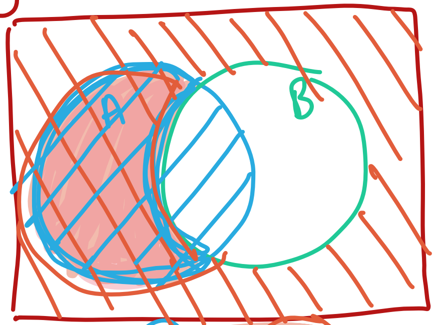
③ take it both weeks

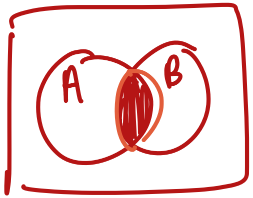
④ take it neither



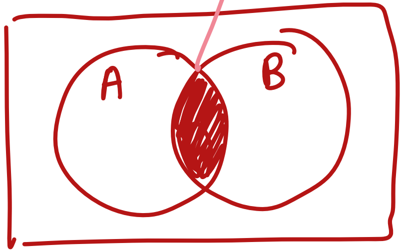
$P(B) = 0.7$

①



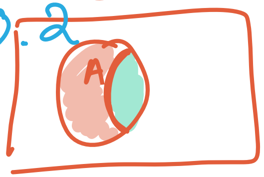


③ $A \cap B$ 0.6



$A = (A \cap B^c) \cup (A \cap B)$

$A \to 0.8$, $A \cap B^c$: 0.2, $A \cap B = 0.6$

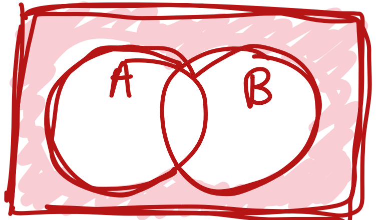


$P(S) = 1$

$P(①) + P(②) + P(③) + P(④) = 1$

$P(④) = 0.1$

④



$P(①) + P(②) + P(③) = 1 - 0.1 = 0.9$

$P(A \cup B) = 0.9$

$= P(A) + P(B) - P(A \cap B)$ with 0.8, 0.7, 0.6