Chapter 3: Independent Events

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Learning objectives

1. Define independence of 2-3 events given probability notation
2. Calculate whether two or more events are independent

Revisiting our coin toss

Question: Which of the following sequences of coin tosses of heads (H) and tails (T) is more likely to happen, assuming the coin is fair?

① → HTTHHHTHTHHTTTH

or

$P(H) = 1/2$

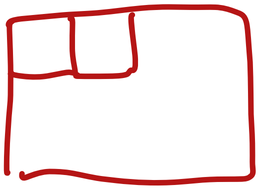
② HTTTTTTTTHTTTTT

seq ① more likely than seq ② ?

1 · 1 · 1 · 1 · 1 · _ · _ · _

2 2 _ _ _ _ _ _ 2

$P(H) = \frac{1}{2}$ $\quad P(①) = \frac{|①|}{|S|} = \frac{(1)^{15}}{(2)^{15}}$ $(1)^{15} = 1$



$P(① \cap ②) = P(①)\,P(②)$

★ come back to this Nicky

$= \left(\frac{1}{2}\right)^{15}$

$P(②) = \frac{|②|}{|S|} = \frac{1^{15}}{2^{15}} = \left(\frac{1}{2}\right)^{15}$

① ⫫ ②

each coin flip is independent

Independent Events

Definition: Independence

Events A and B are **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Notation: For shorthand, we sometimes write

$$A \perp\!\!\!\perp B,$$

$\perp\!\!\!\perp \quad \perp$

to denote that A and B are independent events.

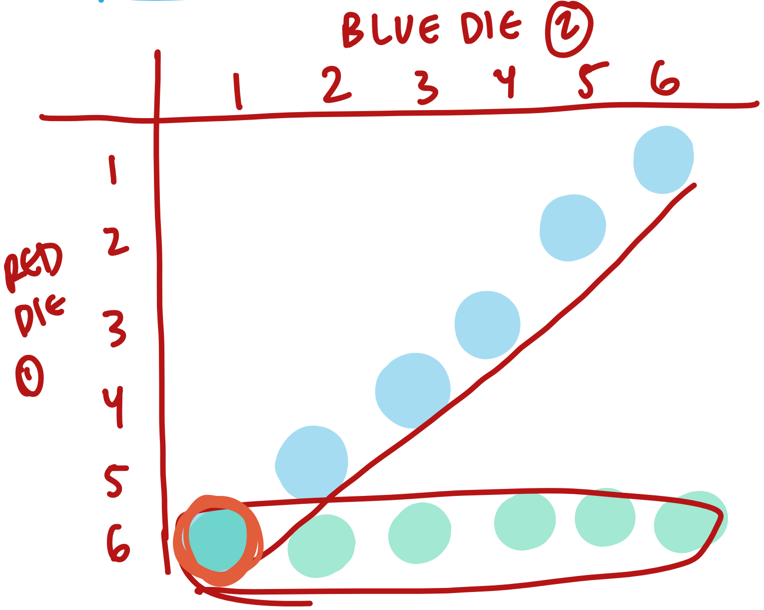
Example of two dice

☆ EX disjoint vs. independent

Example 1

Two dice (red and blue) are rolled. Let A = event a total of 7 appears, and B = event red die is a six. Are events A and B independent?

$$P(A \cap B) = P(A) \cdot P(B)$$



$$P(A) = \frac{\cancel{6}}{6 \times \cancel{6}} = \frac{6}{36} = \frac{1}{6}$$

$$P(B) = \frac{6}{36} = \frac{1}{6}$$

$$P(A \cap B) = \frac{1}{36}$$

$$P(A)P(B) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$$

$$\Rightarrow \underbrace{P(A \cap B)}_{1/36} = \underbrace{P(A)P(B)}_{1/36} \Rightarrow$$

$A \perp B$

A & B are independent

if ind $\Rightarrow$

$P(A \cap B) = P(A)P(B)$

$\Rightarrow$ $P(A \cap B) = P(A)P(B)$ $\Rightarrow$ independent

Independence of 3 Events

Definition: Independence of 3 Events

Events A, B, and C are *mutually* **independent** if

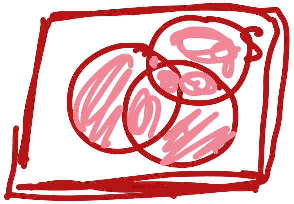
$A \perp B, \ A \perp C, \ B \perp C$

1. - $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$
   - $\mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C)$
   - $\mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C)$
2. $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$

Remark:

On your homework you will show that $(1) \not\Rightarrow (2)$ and $(2) \not\Rightarrow (1)$.

Probability at least one smoker



Example 2

Suppose you take a random sample of n people, of which people are smokers and non-smokers independently of each other. Let

- A_i = event person i is a smoker, for $i = 1, \dots, n$, and
- p_i = probability person i is a smoker, for $i = 1, \dots, n$.

Find the probability that at least one person in the random sample is a smoker.

$$A_i \perp A_j \quad i \neq j \quad i = 1, \dots, n \quad j = 1, \dots, n$$

AXIOM: $P(A) + P(A^c) = 1$

$$\Rightarrow A_i^c \perp A_j^c \qquad P(A_1^c \cap A_2^c) = P(A_1^c)P(A_2^c)$$

De Morgans + complement AXIOM

$$P(\text{at least one smoker}) = P\left(\bigcup_{i=1}^{n} A_i\right)$$

$$= 1 - \left[P\left(\bigcap_{i=1}^{n} A_i^c\right)\right]$$

$$= 1 - \left[\underbrace{P(A_1^c)}_{1-p_1} \cdot \underbrace{P(A_2^c)}_{1-p_2} \underbrace{P(A_3^c)}_{1-p_3} \cdots P(A_n^c)\right]$$

$$= 1 - \prod_{i=1}^{n} (1 - \underbrace{p_i}_{p}) = 1 - (1-p)^n$$

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdot a_3 \cdots a_n$$

$$1-(1-p)^n$$

$$\lim_{n\to\infty} 1-\underbrace{(1-p)}_{1-p\le 1}{}^n = 1$$

$$(1-p)^n \xrightarrow{n\to\infty} 0$$