Chapter 4: Conditional Probability

Meike Niederhausen and Nicky Wakim

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Learning Objectives

1. Use set process to calculate probability of event of interest
2. Calculate the probability of an event occurring, given that another event occurred.
3. Define keys facts for conditional probabilities using notation.

General Process for Probability Word Problems

1. Clearly define your events of interest
2. Translate question to probability using defined events OR Venn Diagram
3. Ask yourself:
   - Are we sampling with or without replacement?
   - Does order matter?
4. Use axioms, properties, partitions, facts, etc. to define the end probability calculation into smaller parts
   - If probabilities are given to you, Venn Diagrams may help you parse out the events and probability calculations
   - If you need to find probabilities with counting, pictures or diagrams might help here
5. Write out a concluding statement that gives the probability context
6. (For own check) Make sure the calculated probability follows the axioms. Is is between 0 and 1?

Let: $A =$
$B =$

$P(A \cap B)$

$C = A \cap B$

$C = A \mid B$

$P(C \cup D) =$
$P(A \cap B)$

Let's revisit our deck of cards

Example 1

Suppose we randomly draw 2 cards from a standard deck of cards. What is the probability that we draw a spade then a heart?

① Let

- Let A = event 1^{st} card is spade
- Let B = event 2^{nd} card is heart

② $P(A \cap B)$
1st Spade, 2nd heart

③ sampling w/ rep? NO
order matter? YES

④ counting: $|A \cap B|$: $|event|$: $13 \cdot 13$
$|S|$: $52 \quad 51$

$$P(A \cap B) = \frac{|A \cap B|}{|S|} = \frac{13 \cdot 13}{52 \cdot 51} = 0.064$$

$$P(A \cap B) = P(A)P(B|A)$$

1 card is drawn

$$\frac{13}{52} \cdot \frac{13}{52}$$

$$P(A \cap B) = P(A)P(B)$$

Conditional Probability facts (1/2)

Fact 1: General Multiplication Rule

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B|A)$$

$P(A \cap B) = P(A) \cdot P(B|A)$

1st spade and heart

Fact 2: Conditional Probability Definition

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

if $A \perp B$, $P(B|A) = P(B)$

$P(A \cap B) = P(B)P(A|B)$

$$\frac{P(A \cap B)}{P(B)} = \frac{P(B)P(A|B)}{P(B)}$$

Conditional Probability facts (2/2)

★ check fact 4 does NOT hold if NOT ind

Fact 3

If A and B are independent events ($A \perp\!\!\!\perp B$), then

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$

DEF of IND $P(A \cap B) = P(A)P(B)$ / $P(B)$

FACT 2 $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(A) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow P(A) = P(A|B)$

Fact 4 if $A \perp B$

$\mathbb{P}(A|B)$ is a probability, meaning that it satisfies the probability axioms. In particular,

$$\mathbb{P}(A|B) + \mathbb{P}(A^C|B) = 1$$

$A \perp B \Rightarrow A^c \perp B$

$P(A|B) = P(A)$ $\quad P(A^c|B) = P(A^c)$

$P(A) + P(A^c) = 1$

$P(A|B) + P(A^c|B) = 1$

□

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

Conditional probability with two dice

Example 2

Two dice (red and blue) are rolled. If the dice do not show the same face, what is the probability that one of the dice is a 1?

A

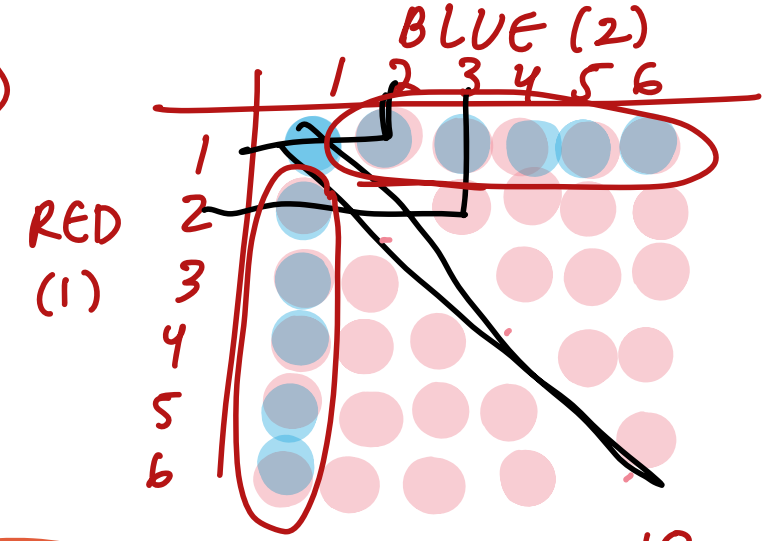
B

① Let A = one die is a 1
B = not same face

② goal: $P(A|B)$

③ replacement? NO
order? NO

④



$$P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow \frac{10}{36}$$

$P(B) \rightarrow \frac{(36-6)}{36}$

$$P(A|B) = \frac{10/36}{30/36} = \frac{1}{3} = \frac{30}{36}$$

$A \perp B$? NO
$P(A) = \frac{11}{36}$
$P(A|B) \neq P(A)$

⑤ The prob of one die showing 1 given dice do not show same face is $\frac{1}{3}$.