Chapter 5: Bayes' Theorem

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Learning Objectives

1. Calculate conditional probability of an event using Bayes' Theorem
2. Utilize additional probability rules in probability calculations, specifically the Higher Order Multiplication Rule and the Law of Total Probabilities

Introduction

- So we learned about conditional probabilities
  - We learned how the occurrence of event A affects event B (B conditional on A)
- Can we figure out information on how the occurrence of event B affects event A?
- We can use the conditional probability ($\mathbb{P}(\mathrm{A}|\mathrm{B})$) to get information on the flipped conditional probability ($\mathbb{P}(\mathrm{B}|\mathrm{A})$)

P(A|B) can help w/ P(B|A)

Bayes' Rule for two events

Theorem: Bayes' Rule (for two events)

For any two events A and B with nonzero probabilties,

$$\mathbb{P}(\mathrm{A}|\mathrm{B}) = \frac{\mathbb{P}(\mathrm{A}) \cdot \mathbb{P}(\mathrm{B}|\mathrm{A})}{\mathbb{P}(\mathrm{B})}$$

$$P(A \cap B) = P(A)\, P(B \mid A)$$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Calculating probability with Higher Order Multiplication Rule

$P(A \cap B)$

Example 1

Suppose we draw 5 cards from a standard shuffled deck of 52 cards. What is the probability of a flush, that is all the cards are of the same suit (including straight flushes)?

Higher Order Multiplication Rule

$$\mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2 | A_1) \cdot \mathbb{P}(A_3 | A_1 A_2) \ldots \cdot \mathbb{P}(A_n | A_1 A_2 \ldots A_{n-1})$$

$P(A \cap B \cap C) = P(A)P(B|A) \cdot P(C|A,B)$

① A_i = get same suit as A_1 $(2 \le i \le 5)$

A_1 = get card of any suit

② $P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5)$

③ ORDER MATTERS
NO REP.

④ $P(A_1) = \frac{52}{52}$ $\quad P(A_2 | A_1) = \frac{12}{51}$ $\quad P(A_3 | A_1, A_2) = \frac{11}{50}$

$P(A_4 | A_1, A_2, A_3) = \frac{10}{49}$ $\quad P(A_5 | A_1, A_2, A_3, A_4) = \frac{9}{48}$

$$P(A_1 \cap A_2 \cap \ldots \cap A_5) = \frac{52 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = 0.00198$$

⑤ The probability of a flush is 0.00198.

Calculating probability with Law of Total Probability

Example 2 % CB / SAB

Suppose 1% of people assigned female at birth (AFAB) and 5% of people assigned male at birth (AMAB) are color-blind. Assume person born is equally likely AFAB or AMAB (not including intersex). What is the probability that a person chosen at random is color-blind?

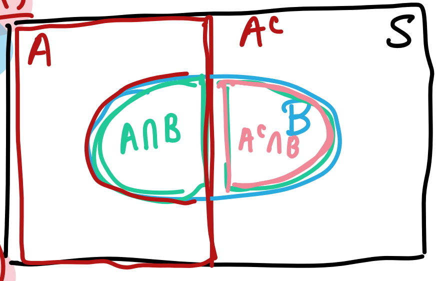
Law of Total Probability for 2 Events

For events A and B,

$$\mathbb{P}(\mathrm{B}) = \mathbb{P}(\mathrm{B} \cap \mathrm{A}) + \mathbb{P}(\mathrm{B} \cap \mathrm{A}^C)$$
$$= \mathbb{P}(\mathrm{B}|\mathrm{A}) \cdot \mathbb{P}(\mathrm{A}) + \mathbb{P}(\mathrm{B}|\mathrm{A}^C) \cdot \mathbb{P}(\mathrm{A}^C)$$

sum of disjoint events

$P(B|A) = \frac{P(B \cap A)}{P(A)}$



① A = AFAB
A^c = AMAB
B = person is colorblind

$B|A$

② $P(B)$?

③ order/rep? N/A

④ $P(B|A) = 0.01$
$P(B|A^c) = 0.05$

$P(B) = P(B|A)P(A) + P(B|A^c)P(A^c)$

prob of AFAB @ birth

AMAB

$B = ($

$P(B) = 0.01(0.5) + 0.05(0.5)$

$P(B) = 0.03$

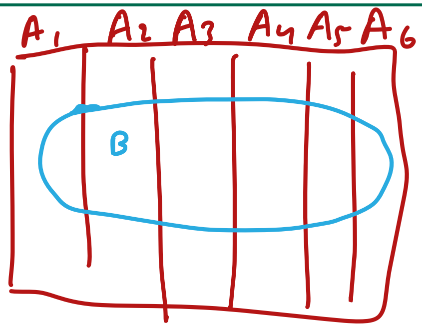
⑤ The probability that person is CB is 0.03.

General Law of Total Proability

Law of Total Probability (general)

If $\{A_i\}_{i=1}^{n} = \{A_1, A_2, \ldots, A_n\}$ form a partition of the sample space, then for event B,

$$\begin{aligned} \mathbb{P}(B) &= \sum_{i=1}^{n} \underline{\mathbb{P}(B \cap A_i)} \\ &= \sum_{i=1}^{n} \underline{\mathbb{P}(B|A_i) \cdot \mathbb{P}(A_i)} \end{aligned}$$



Calculating probability with generalized Law of Total Probability

Example 3

Individuals are diagnosed with a particular type of cancer that can take on three different disease forms,* D_1, D_2, and D_3. It is known that amongst people diagnosed with this particular type of cancer,

- 20% of people will eventually be diagnosed with form D_1,
- 30% with form D_2, and
- 50% with form D_3.

The probability of requiring chemotherapy (C) differs among the three forms of disease:

- 80% with D_1,
- 30% with D_2, and
- 10% with D_3.

Based solely on the preliminary test of being diagnosed with the cancer, what is the probability of requiring chemotherapy (the event C)?

at home

Let's revisit the color-blind example

Example 4

Recall the color-blind example (Example 2), where

- a person is AMAB with probability 0.5, $P(A^c)$
- AMAB people are color-blind with probability 0.05, and $P(B|A^c)$
- all people are color-blind with probability 0.03. $P(B)$

Assuming people are AMAB or AFAB, find the probability that a color-blind person is AMAB. $P(A^c|B)$

① - ③ @ home

④ $P(A^c|B) = \dfrac{P(A^c \cap B)}{P(B)} = \dfrac{P(B|A^c)\,P(A^c)}{P(B)}$

$= \dfrac{(0.05)(0.5)}{0.03}$

$= 0.83\overline{3}$

⑤ The probability that person is AMAB given they are color blind is $0.83\overline{3}$.

Calculate probability with both rules

★ go back to write explanation

Example 5

Suppose

- 1% of women aged 40-50 years have breast cancer, → $P(B) = 0.01$
- a woman with breast cancer has a 90% chance of a positive test from a mammogram, and $P(A|B) = 0.9$
- a woman has a 10% chance of a false-positive result from a mammogram. → does NOT have BC $P(A|B^c) = 0.1$

What is the probability that a woman has breast cancer given that she just had a positive test?

① S = woman aged 40-50 yrs old
→ A = positive test for BC
→ B = has breast cancer

② $P(B|A) = ?$ ③ N/A

④ gen. mult rule

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A|B)P(B)}{P(A)} ★$$

law of total prob

$$P(A \cap B)$$

$$= \frac{P(A|B)P(B)}{\underbrace{P(A|B)P(B)}_{(P(A \cap B))} + \underbrace{P(A|B^c)P(B^c)}_{(P(A \cap B^c))}}$$

$$= \frac{0.9 \cdot 0.01}{0.9 \cdot 0.01 + 0.1 \cdot \underbrace{(1 - 0.01)}_{P(B^c) = 1 - P(B)}}$$

$$= 0.833$$

⑤ The probability that a woman aged 40-50 yrs old has BC given positive test is 0.833.

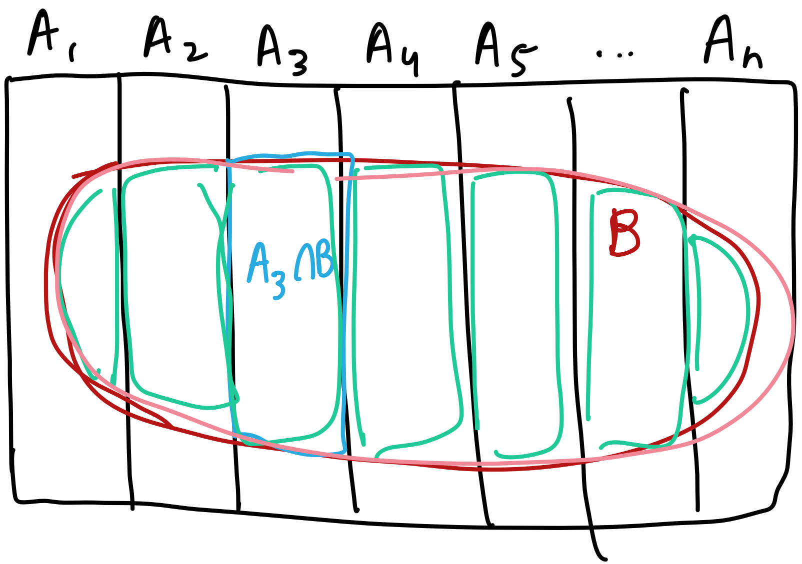
Bayes' Rule

Theorem: Bayes' Rule

If $\{A_i\}_{i=1}^{n}$ form a partition of the sample space S, with $\mathbb{P}(A_i) > 0$ for $i = 1 \ldots n$ and $\mathbb{P}(B) > 0$, then

$$\mathbb{P}(A_j|B) = \frac{\mathbb{P}(B|A_j) \cdot \mathbb{P}(A_j)}{\sum_{i=1}^{n} \mathbb{P}(B|A_i) \cdot \mathbb{P}(A_i)}$$

$\sum_{i=1}^{n} \mathbb{P}(B|A_i) \cdot \mathbb{P}(A_i) \rightarrow P(B)$



$$P(A_3|B) = \frac{P(A_3 \cap B)}{P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) + \ldots + P(A_n \cap B)}$$

1% have BC $\rightarrow P(BC) = 0.01$

90% w/ BC have pos. test $\rightarrow P(pos/BC)$

15% chance of $\underbrace{\text{false pos.}}_{pos/BC^c}$ $\rightarrow P(pos/BC^c)$

	P	N
BC	$BC \cap P$	$BC \cap N$
BC^c	$BC^c \cap P$	$BC^c \cap N$

$P(P/BC) = 0.9 = \dfrac{BC \cap P}{BC \cap P + BC \cap N}$

$P(BC/P) = \dfrac{P(BC \cap P)}{P(BC \cap P) + P(BC^c \cap P)}$

$\underline{P(BC/pos)}$

$$P(BC/pos) = \frac{P(BC \text{ and } pos)}{P(positive)}$$

$$= \frac{P(pos/BC) \cdot P(BC)}{P(pos \cap BC) + P(pos \cap BC^c)}$$

$=$