Chapter 9: Independence and Conditioning (Joint Distributions)

Meike Niederhausen and Nicky Wakim

2023-10-11

Table of contents

Learning Objectives

1. Calculate probabilities for a pair of discrete random variables
2. Calculate ~~and graph~~ a *joint*, *marginal*, *and conditional* probability mass function (pmf)
3. Calculate ~~and graph~~ a *joint*, *marginal*, *and conditional* cumulative distribution function (CDF)

What is a joint pmf?

X_1 & X_2 ... X_n

Definition: joint pmf

The **joint pmf** of a pair of discrete r.v.'s X and Y is

$$p_{X,Y}(x, y) = \mathbb{P}(X = x \text{ and } Y = y) = \mathbb{P}(X = x, Y = y)$$

$p_X(x)$ $\quad$ $p_Y(y)$

This chapter's main example

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

1. Find $p_{X,Y}(x, y)$.
2. Find $\mathbb{P}(X + Y = 3)$.

joint pmf

3. Find $\mathbb{P}(Y = 1)$.
4. Find $\mathbb{P}(Y \leq 2)$.

marginal pmf

5. Find the joint CDF $F_{X,Y}(x, y)$ for the joint pmf $p_{X,Y}(x, y)$
6. Find the marginal CDFs $F_X(x)$ and $F_Y(y)$

joint CDF

7. Find $p_{X|Y}(x|y)$.
8. Are X and Y independent? Why or why not?

conditional pmf

Joint pmf

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

1. Find $\boxed{p_{X,Y}(x, y).}$ ✓
2. Find $\mathbb{P}(X + Y = 3)$.

①

X (1st draw) \ Y (2nd draw)	1	2	3
1	0	1/6	1/6
2	1/6	0	1/6
3	1/6	1/6	0

draw $X = 1, Y = 1$

↳ impossible

$\rightarrow P(X=1, Y=1) = 0$

draw $X = 1$

$P(Y=2 \mid X=1) = \frac{1}{2}$

$P(Y=3 \mid X=1) = \frac{1}{2}$

$\rightarrow P(X=1, Y=2) = P(X=1)P(Y=2 \mid X=1)$

$P(A \cap B) = P(A)P(B \mid A)$

$= \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

$P(X=1, Y=3) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

so what is $p_{X,Y}(x,y)$?

$$p_{X,Y}(x,y) = \begin{cases} 1/6 & x \neq y \\ 0 & x = y \end{cases}$$

for $x = 1, 2, 3$ & $y = 1, 2, 3$ (OR $x, y = 1, 2, 3$)

② $P(X+Y=3) = \sum_{\{x+y=3\}}\sum p_{X,Y}(x,y)$

when does $X+Y=3$

$P(X+Y=3) = p_{X,Y}(1,2) + p_{X,Y}(2,1)$

$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

Marginal pmf's

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

3. Find $\mathbb{P}(Y = 1)$.

4. Find $\mathbb{P}(Y \le 2)$.

		Y (2nd draw) 1	2	3	$P_X(x)$
X (1st draw)	1	0	1/6	1/6	1/3 $P(X=1)$
	2	1/6	0	1/6	1/3
	3	1/6	1/6	0	1/3
$P_Y(y)$		1/3 $P(Y=1)$	1/3	1/3	

$\left.\begin{array}{l} P_Y(y) \\ \underbrace{P_X(x)} \end{array}\right\}$ marginal pmf

↳ sum across all Y values

③ $P(Y=1) = P(X=1, Y=1) + P(X=2, Y=1) + P(X=3, Y=1)$

$= P_{X,Y}(1,1) + P_{X,Y}(2,1) + P_{X,Y}(3,1) = \sum_{y=1}^{1} \sum_{x=1}^{3} P_{X,Y}(x,y)$

$= 0 + \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

④ $P(Y \le 2) = P(Y=1) + P(Y=2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

Remarks on the joint pmf

Some properties of joint pmf's:

- A joint pmf $p_{X,Y}(x, y)$ must satisfy the following properties:
  - $p_{X,Y}(x, y) \geq 0$ for all x, y.
  - $\sum_{\{\text{all } x\}} \sum_{\{\text{all } y\}} p_{X,Y}(x, y) = 1.$
- Marginal pmf's:
  - $p_X(x) = \sum_{\{\text{all } y\}} p_{X,Y}(x, y)$
  - $p_Y(y) = \sum_{\{\text{all } x\}} p_{X,Y}(x, y)$

What is a joint CDF?

Definition: joint CDF

The **joint CDF** of a pair of discrete r.v.'s X and Y is

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x \text{ and } Y \leq y) = \mathbb{P}(X \leq x, Y \leq y)$$

Joint CDFs

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

5. Find the joint CDF $F_{X,Y}(x, y)$ for the joint pmf $p_{X,Y}(x, y)$

jPMF

X (1st draw) \ Y (2nd draw)	1	2	3	$P_X(x)$
1	0	1/6	1/6	1/3
2	1/6	0	1/6	1/3
3	1/6	1/6	0	1/3
$P_Y(y)$	1/3	1/3	1/3	

jCDF

X \ Y	1	2	3
1	0	1/6	1/3
2	1/6	1/3	2/3
3	1/3	2/3	3/3

$P(X \le 1, Y \le 1)$

$= P(X=1, Y=1) = 0$

$P(X \le 1, Y \le 2) = P(X=1, Y=1) + P(X=1, Y=2) = 0 + \frac{1}{6}$

$P(X \le 1, Y \le 3) = P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3) = 0 + \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$P(X \le 3, Y \le 2) = P_{X,Y}(1,1) + P_{X,Y}(1,2) + P_{X,Y}(2,1) + P_{X,Y}(2,2)$

$F_{X,Y}(3,2)$ $\quad + P_{X,Y}(3,1) + P_{X,Y}(3,2) = 0 + \frac{1}{6} + \frac{1}{6} + 0 + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$

Marginal CDFs

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

6. Find the marginal CDFs $F_X(x)$ and $F_Y(y)$

jCDF

X \ Y	1	2	3	$F_X(x)$
1	0	1/6	1/3	1/3
2	1/6	1/3	2/3	2/3
3	1/3	2/3	3/3	$\boxed{3/3}$
$F_Y(y)$	1/3	2/3	3/3	

jPMF

X (1st draw) \ Y (2nd draw)	1	2	3	$P_X(x)$
1	0	1/6	1/6	1/3
2	1/6	0	1/6	1/3
3	1/6	1/6	0	1/3
$P_Y(y)$	1/3	1/3	1/3	

FOR NICKY

★ double check

⑥ $F_X(x)$

$F_X(1) = P(X \leq 1) = P_X(1) = 1/3$

$= P_{X,Y}(1,1) + P_{X,Y}(1,2) + P_{X,Y}(1,3)$

$F_X(2) = P(X \leq 2) = P(X=1) + P(X=2) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

$F_X(3) = P(X \leq 3) = P(X=1) + P(X=2) + P(X=3) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3}$

piecewise:

$$F_X(x) = \begin{cases} 0 & x < 1 \\ 1/3 & 1 \leq x < 2 \\ 2/3 & 2 \leq x < 3 \\ 3/3 & x \geq 3 \end{cases}$$

Remarks on the joint and marginal CDF

- $F_X(x)$: right most columns of the joint CDF table (where the Y values are largest)
- $F_Y(y)$: bottom row of the table (where X values are largest)

within CDF diagram

- $F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y)$ $F_{X,Y}(x, y \to \infty)$
- $F_Y(y) = \lim_{x\to\infty} F_{X,Y}(x,y)$

Independence and Conditioning

Recall that for *events* A and B,

- $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$
- A and B are independent if and only if
  - $\mathbb{P}(A|B) = \mathbb{P}(A)$
  - $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

Independence and conditioning are defined similarly for r.v.'s, since

$$\underbrace{p_X(x) = \mathbb{P}(X = x)}_{P(X)} \text{ and } \underbrace{p_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y)}_{P(X \cap Y)}.$$

if ind $\quad p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$

What is the conditional pmf?

Definition: conditional pmf $P(A|B)$

The **conditional pmf** of a pair of discrete r.v.'s X and Y is defined as

$$p_{X|Y}(x|y) = \mathbb{P}(X = x|Y = y) = \frac{\mathbb{P}(X = x \text{ and } Y = y)}{\mathbb{P}(Y = y)} = \frac{p_{X,Y}(x,y)}{p_Y(y)} \quad \frac{\text{joint}}{\text{marg.}}$$

$\boxed{\text{if } p_Y(y) > 0.}$

$$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$$

Remarks on the conditional pmf

The following properties follow from the conditional pmf definition:

- If $X \perp Y$ (independent)
  - $p_{X|Y}(x|y) = p_X(x)$ for all x and y
  - $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ for all x and y
  - Which also implies (⇒): $F_{X,Y}(x, y) = F_X(x)F_Y(y)$ for all x and y
- If $X_1, X_2, \ldots, X_n$ are independent random variables (1-n)
  - $$p_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{i=1}^{n} p_{X_i}(x_i)$$
    $p_{X_1}(x_1) \cdot p_{X_2}(x_2) \cdot \ldots p_{X_n}(x_n)$
  - $$F_{X_1,X_2,\ldots,X_n}(x_1, x_2, \ldots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n) = \prod_{i=1}^{n} P(X_i \le x_i) = \prod_{i=1}^{n} F_{X_i}(x_i)$$
    $= F_{X_1}(x_1) \cdot F_{X_2}(x) \cdots F_{X_n}(x_n)$

Conditional pmf's

Example 1

Let X and Y be two random draws from a box containing balls labelled 1, 2, and 3 without replacement.

7. Find $p_{X|Y}(x|y)$.
8. Are X and Y independent? Why or why not?

Remark:

- To show that X and Y are *not* independent, we just need to find one counter example
- However, to show that they *are* independent, we need to verify this for all possible pairs of x and y

another way $p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$

⑦

		Y (2nd draw)			
		1	2	3	$p_X(x)$
X (1st draw)	1	~~0~~	1/6	1/6	1/3
	2	1/6	0	1/6	1/3
	3	1/6	1/6	0	1/3
$p_Y(y)$		1/3	1/3	1/3	

⑦ $p_{X|Y}(x|y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)}$

$p_{X|Y}(2|1) = \dfrac{1/6}{1/3} = \boxed{\dfrac{1}{2}}$

$p_{X|Y}(3|2) = \dfrac{p_{X,Y}(3,2)}{p_Y(2)} = \dfrac{1/6}{1/3} = \dfrac{1}{2}$

$p_{X|Y}(3|3) = \dfrac{p_{X,Y}(3,3)}{p_Y(3)} = \dfrac{0}{1/3} = 0$

$p_{X|Y}(x|y) = \begin{cases} 1/2 & x \neq y \\ 0 & x = y \end{cases}$ for $x, y = 1, 2, 3$

IF IND

⑧ $\underline{p_{X|Y}(x|y)} = p_X(x)$ $\quad \underline{x=1}, y=2$

$1/2 \neq 1/3$ $\quad \boxed{X \not\perp Y \text{ not ind}}$