Chapter 10: Expected Values of Discrete RVs

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2023-10-18

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Learning Objectives

1. Calculate the mean (expected value) of discrete random variables

Our good and fair friend, the 6-sided die

Example 1

Suppose you roll a fair 6-sided die. What value do you expect to get?

$\frac{1}{6}$ prob for each side

$$\text{avg} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6}$$

$$\sum_{i=1}^{6} x$$

$$= 3.5$$

$$\underline{\text{weighted}}\ \text{avg} = \left(\frac{1}{6}\right)1 + \left(\frac{1}{6}\right)2 + \left(\frac{1}{6}\right)3 + \left(\frac{1}{6}\right)4 + \left(\frac{1}{6}\right)5 + \left(\frac{1}{6}\right)6 = 3.5$$

What is an expected value?

$x_1, x_2, \dots, x_n$

$\left[X_1, X_2, \dots X_n \right.$
$\left. \hookrightarrow n \text{ RVs} \right]$

Definition: Expected value

The **expected value** of a discrete r.v. X that takes on values $x_1, x_2, \dots, x_n$ is

$$\mathbb{E}[X] = \sum_{i=1}^{n} x_i p_X(x_i).$$

- Expected values are not necessarily an actual outcome
  - In previous example, we cannot roll a 3.5
  - It could be that our expected value is not in the sample space ($E(X) \notin S$)
- Definition holds when X takes on countably infinitely many values (think $n = \infty$)

$$E[X] = \sum_{i=1}^{\infty} x_i p_X(x_i)$$

Our good and not-so-fair friend, the 6-sided die

Example 2

Suppose the die is 6-sided, but not fair. And the probabilities of each side is distributed as:

x	$p_X(x)$
1	0.10
2	0.05
3	0.02
4	0.30
5	0.50
6	0.03

What value do you expect to get on a roll?

$$E[X] = \sum_{i=1}^{6} x_i \, p_X(x_i)$$

$$= 1(0.10) + 2(0.05) + 3(0.02) + 4(0.30) + 5(0.50) + 6(0.03)$$

$$= 4.14$$

☆ still weighted avg (weight = prob

☆ DO NOT round expected value to nearest whole #.

Expected value of a Bernoulli distribution

Example 3

Suppose

$$X = \begin{cases} 1 & \text{with probability } p \quad \text{(success)} \\ 0 & \text{with probability } 1-p \quad \text{(failure)} \end{cases}$$

$n = 2$

Find the expected value of X.

$$E[X] = \sum_{i=1}^{n} x_i P_X(x_i) = \underbrace{0(1-p)}_{0} + \underbrace{1(p)}_{p} = p$$

$E(X)$

$[\sum \sim]$

Let's slightly change our random variable

Example 5

Suppose

$$X = \begin{cases} 1 & \text{with probability } p \\ -1 & \text{with probability } 1-p \end{cases}$$

Find the expected value of X.

$$E[X] = \sum_{i=1}^{2} x_i \, p_X(x_i) = (1)(p) + (-1)(1-p) = p - 1 + p = \underline{2p-1}$$

$p = 1/2$ $\quad 2p-1 = 2(\frac{1}{2}) - 1 = 0$ $\quad$ fair coin = no gain no loss over time

$p > 1/2$ $\quad 2p-1 > 0$ $\quad$ gain over time

$p < 1/2$ $\quad 2p-1 < 0$ $\quad$ loss over $\boxed{\text{many coin flips}}$

Ghost!

Example 6

A ghost is trick-or-treating. It comes to a house where it is known that there are 30 candies in the bag and only one is a watermelon Jolly Rancher, which is the ghost's favorite. The ghost takes pieces of candy without replacement until it gets the watermelon Jolly Rancher. How many pieces of candy do we expect the ghost to take?

Let X = # draws to get WJR

DOUBLE CHECK

$\binom{29}{3}\binom{1}{1}\binom{26}{26}$

What is $\boxed{p_X(x)?}$

intuitively: $\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{\overset{WJR}{\_}}\ \underset{5}{\_}\ \dots\ \underset{29}{\_}\ \underset{30}{\_}$

$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{\_}\ \underset{5}{\_}\ \dots\ \underset{29}{\overset{WJR}{\_}}\ \underset{30}{\_}$

1/30

directly: $P(X=1) = \frac{1}{30}$

$P(X=4) = \frac{\cancel{29}}{30}\cdot\frac{\cancel{28}}{\cancel{29}}\cdot\frac{\cancel{27}}{\cancel{28}}\cdot\frac{1}{\cancel{27}} = \frac{1}{30}$

$\frac{26}{26}\cdot\frac{25}{25}\cdots$

$P(X=2) = \frac{\cancel{29}}{30}\cdot\frac{1}{\cancel{29}} = \frac{1}{30}$

$P(X=j) = \frac{1}{30}$ for $j = 1, 2, 3, \dots, 30$

$$E[X] = \sum_{i=1}^{30} x_i p_X(x_i) = \sum_{i=1}^{30} x_i \left(\frac{1}{30}\right)$$

if w/ replacement

$P(X=1) = \frac{1}{30}$

$P(X=2) = \rightarrow$

$P(X=3) -$

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$$= \frac{1}{30} \sum_{i=1}^{30} x_i$$

$$= \frac{1}{30}(1+2+3+\ldots+30)$$

$$= \frac{1+2+3+\ldots+30}{30}$$

$$= \frac{465}{30} = 15.5$$

We expect the ghost to take 15.5 candies until it gets the WJR.