Chapter 11: Expected Values of Sums of Discrete RVs

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Learning Objectives

1. Calculate the mean (expected value) of *sums of* discrete random variables

Revisiting our two card draw

Example 1

Suppose you draw 2 cards from a standard deck of cards *with* replacement. Let X be the number of hearts you draw. Find $\mathbb{E}[X]$.

Recall Binomial RV with $n = 2$:

$$p_X(x) = \binom{2}{x} p^x (1-p)^{2-x} \quad \text{for } x = 0, 1, 2$$

x	0	1	2
$p_X(x)$	$\cancel{\binom{2}{0}}\cancel{\left(\frac{1}{4}\right)^0}\left(\frac{3}{4}\right)^2$	$\binom{2}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)$	$1\cdot\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)$

$P(\heartsuit) = \frac{13}{52} = \frac{1}{4} \qquad P(\heartsuit^c) = \frac{3}{4}$

$$E[X] = \sum_{i=1}^{3} x_i \, p_X(x_i)$$

$\rightarrow \sum_{\text{all } x} p_X(x_i) = 1$

$$= 0\left(\frac{3}{4}\right)^2 + (1)\left[2\cdot\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\right] + 2\cdot 1\cdot\left(\frac{1}{4}\right)^2$$

$$= 0 + \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

We expect to get 1/2 hearts

What if we draw A LOT of cards?

Example 2

What is the expected number of hearts in Example 1 if you draw 200 cards?

Recall Binomial RV with $n = 200$:

$$p_X(x) = \binom{200}{x} p^x (1-p)^{200-x}$$

for $x = 0, 1, 2, \ldots, 200$

0-200

$x = -1,\ 3,\ 6,\ 20$

$X_1\quad X_2\quad X_3\quad X_4$

$i = 1,\ 2,\ 3,\ 4$

X = # of ♡s from 200 cards

$$E[X] = \sum_{x=0}^{200} x\, p_X(x)$$

$$= \sum_{x=0}^{200} x \left[\binom{200}{x} p^x (1-p)^{200-x} \right]$$

★ Binomial is the SUM of n bernoullis

So let $Y_i = \begin{cases} 1 & i\text{th card is } ♡ \\ 0 & \text{all else} \end{cases}$

for $i = 1, 2, 3, \ldots, 200$

$$\underset{\text{binom.}}{X} = \sum_{i=1}^{200} Y_i$$

$$\boxed{E[X] = E\left[\sum_{i=1}^{200} Y_i\right]}$$

Sum of discrete RVs

$X_1, X_2, \dots, X_n$ n RVs.

Theorem 11.1: Sum of discrete RVs

For discrete r.v.'s X_i and constants a_i, $i = 1, 2, \dots, n$,

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i].$$

constant for ex: 3 is a constant

$$E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E(X_i)$$

Remark: The theorem holds for infinitely r.v.'s X_i as well.

- For two RVs, X and Y: $(X_1 \;\&\; X_2)$
  - We can say $E[X + Y] = E[X] + E[Y]$
  - ... and constant numbers a and b, we can also say $E[aX + bY] = aE[X] + bE[Y]$
  - We can also also say $E[X - Y] = E[X] - E[Y]$, since $b = -1$

$X + (-1)Y$

Corollaries from Thm 11.1

IF BOTH BERN w/ p

$E(X+Y) = E(X) + E(Y) = 2E(X)$

Corollary 11.1.1

For a discrete r.v. X, and constants a and b,

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b.$$

$E(a) = a$

$E(3) = 3$

$E(aX) = aE(X)$

$E(3X) = 3E(X)$

Corollary 11.1.2

If $X_i, i = 1, 2, \ldots, n$, are identically distributed r.v.'s, then

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = n\mathbb{E}[X_1].$$

For 200 cards:

$E\left[\sum_{i=1}^{200} X_i\right] = 200 \cdot E[X_1]$

2
3
⋮
200

$= 200 \cdot \left(\frac{1}{4}\right)$ (p)

n

$= 50$

$\boxed{E(\text{Binom}) = np}$

Revisiting our ghost!

Example 3

The ghost is trick-or-treating at a different house now. In this case it is known that the bag of candy has 10 chocolates, 20 lollipops, and 30 laffy taffies. The ghost grabs a handful of five pieces of candy. How many pieces of chocolate do we expect the ghost to take?

X = # chocolate in handful of 5

60 candies

Find $E(X)$

$X = \sum_{i=1}^{5} Y_i$ where $Y_i = \begin{cases} 1 & \text{ith candy is choco} \\ 0 & \text{else} \end{cases}$ for $i = 1, 2, 3, 4, 5$

p

$p(\text{choco}) = \frac{10}{60} = \frac{1}{6}$

$$E[X] = E\left[\sum_{i=1}^{5} Y_i\right] = \sum_{i=1}^{5} E(Y_i) = 5 \cdot \left(\frac{1}{6}\right) = \frac{5}{6}$$

★ really explain diff of w/ & w/out replacement.

X = # choc in 5 draws

$$p_X(x) = \frac{\binom{10}{x}\binom{50}{5-x}}{\binom{60}{5}} \quad \text{for } x = 0, 1, \ldots, 5$$

$$p_X(0) = \frac{\binom{10}{0}\binom{50}{5}}{\binom{60}{5}} = \frac{\left(\frac{50!}{5!\,45!}\right)}{\binom{60}{5}} = \frac{\frac{\overset{10}{\cancel{50}}\cdot 49\cdot 48\cdot 47\cdot 46}{\cancel{5}\cdot\cancel{5}\cdot 4\cdot 3\cdot 2}}{\binom{60}{5}} = \frac{2118760}{\binom{60}{5}}$$

$$p_X(1) = \frac{\binom{10}{1}\binom{50}{4}}{\binom{60}{5}} = \longrightarrow \frac{10!}{1!\,9!}\cdot\frac{50!}{4!\,46!} = \frac{10\cdot 50\cdot 49\cdot 48\cdot 47}{4\cdot 3\cdot 2} = 2303000$$

Cost of hotel rooms

Example 4

A tour group is planning a visit to the city of Minneapolis and needs to book 30 hotel rooms. The average price of a room is \$200. In addition, there is a 10% tourism tax for each room. What is the expected cost for the 30 hotel rooms?