Chapter 18: Poisson RVs

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Learning Objectives

1. Identify the variable and the parameters of a Poisson distribution in a word problem, and state what the variable and parameters mean.
2. Use the formulas for the pmf/CDF, expected value, and variance to answer questions and find probabilities.

Properties of Poisson RVs

- **Scenario:** We are counting the number of successes in a fixed time period, which has a constant rate of successes

spatial

- Shorthand: $X \sim \text{Poisson}(\lambda)$ or $X \sim \text{Pois}(\lambda)$

$$X = \text{Number of successes in a given period}$$

$$p_X(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} \text{ for } x = 0, 1, 2, 3, \ldots$$

$$E(X) = \lambda$$

$$Var(X) = \lambda$$

Distingushing between Binomial and Poisson RVs

- Recall that if $X \sim \text{Binomial}(n, p)$, then
  - X models the number of successes …
  - in n independent (Bernoulli) trials …
  - that each have the same probability of success p.
- Poisson r.v.'s are similar,
  - except that instead of having n discrete independent trials,
  - there is a **fixed time period** during which the successes happen.

time in Poisson (instead of trials)
to count our successes.

Examples of Poisson RVs

- Number of visitors to an emergency room in an hour during a weekend night
- Number of study participants enrolled in a study per week
- Number of pedestrians walking through a square mile
- Any more?

$\lambda = 5$/night

#/hour

$\lambda = 10$/week

spatial

$\lambda = 100$/sq mile per hour

Emergency Room Visitors

Example 1

Suppose an emergency room has an average of 50 visitors per day. Find the following probabilities.

1. Probability of 30 visitors in a day.
2. Probability of 8 visitors in an hour.
3. Probability of at least 8 visitors in an hour.

① X = # visitors / day

$\lambda = 50$ visitors/day

$$P(X=30) = \frac{e^{-50}\,50^{30}}{30!}$$

$$\boxed{P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad \text{for } x = 0,1,2,3,\ldots}$$

② 8 visitors / hour

Y = # visitors / hour

$$P(Y=8) = \frac{e^{-50/24}\left(\frac{50}{24}\right)^{8}}{8!}$$

$$\lambda = \frac{50 \text{ vis.}}{\cancel{\text{day}}} \cdot \frac{1\ \cancel{\text{day}}}{24 \text{ hours}}$$

$= 50/24$ visitors/hr

$= 2.08\overline{3}$ vis/hr

③ $$P(Y \geq 8) = \sum_{y=8}^{\infty} P_Y(y)$$

$$= 1 - P(Y \leq 7)$$

$$= 1 - \sum_{y=0}^{7} P_Y(y) = 1 - \sum_{y=0}^{7} \frac{e^{-50/24}\left(\frac{50}{24}\right)^{y}}{y!}$$

WARNING: ☆ double check

$\underset{y}{8} \cdot \underset{hr}{24} = \underset{x}{192}$

$P(Y=8) \neq P(X=192)$

w/ $\lambda = 2.08$/hr $\qquad \lambda = 50$/day

$e^{-2.08}$ not linear w/ e^{-50}

Combining independent Poisson distributions

Theorem 1

If $X \sim \text{Poiss}(\lambda_1)$ and $Y \sim \text{Poiss}(\lambda_2)$ are independent of each other, then $Z = X + Y \sim \text{Poiss}(\lambda_1 + \lambda_2)$.

$\boxed{\star \text{ does it work w/ } Z = X - Y\text{??}}$

Two emergency rooms

Example 2

Suppose emergency room 1 has an average of 50 visitors per day, and emergency room 2 has an average of 70 visitors per day, independently of each other. What is the probability distribution to model of the total number of visitors to both?

Poisson Approximation of the Binomial

Both Poisson and Binomial r.v.'s are counting the number of successes

- If for a Binomial r.v.
  - the number of trials n is very large, and
  - the probability of success p is close to 0 or 1,
- then the Poisson distribution can be used to approximate Binomial probabilities
  - and we use $\lambda = np$

Rule of Thumb:

$$\frac{1}{10} \leq np(1-p) \leq 10$$

npq

Medical lab errors

Example 3

Suppose that in the long run, errors in a medical testing lab are made 0.1% of the time. Find the probability that fewer than 4 mistakes are made in the next 2,000 tests.

1. Find the probability using the Binomial distribution.
2. Approximate the probability in part (1) using the Poisson distribution.