Chapter 22: Introduction to Counting

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Class Overview

- Basic Counting Examples
- Permutations and Combinations
- More Examples: order matters vs. not

Learning Objectives

1. Define permutations and combinations
2. Characterize difference between sampling with and without replacement
3. Characterize difference between sampling when order matters and when order does not matter
4. Calculate the probability of sampling any combination of the following: *with or without replacement* and *order does or does not matter*

Basic Counting Examples

Basic Counting Examples (1/3)

Example 1

Suppose we have 10 (distinguishable) subjects for study.

1. How many possible ways are there to order them?
2. How many ways to order them if we can reuse the same subject and
   - need 10 total?
   - need 6 total?
3. How many ways to order them *without replacement* and only need 6?
4. How many ways to choose 6 subjects without replacement if the order doesn't matter?

Basic Counting Examples (2/3)

Suppose we have 10 (distinguishable) subjects for study.

$P(A) = \frac{|A|}{|S|}$

Example 1.1

How many possible ways are there to order them?

NO REPLACEMENT
ORDER MATTERS

$\underline{10} \times \underline{9} \times \underline{8} \times \underline{7} \times \underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1}$

1 2 3 4 5 10

$= 10!$

$!$: factorial

$= 3\,628\,800$

10 9

3 → 9

2 → 9

Example 1.2

How many ways to order them if we can reuse the same subject and

- need 10 total?
- need 6 total?

Replacement

$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^{10}$

$10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 10^{6}$

Basic Counting Examples (3/3)

$10!$

Suppose we have 10 (distinguishable) subjects for study.

Example 1.3

How many ways to order them without replacement and only need 6?

$$\underline{10} \cdot \underline{9} \cdot \underline{8} \cdot \underline{7} \cdot \underline{6} \cdot \underline{5} \neq \underline{10!}$$

$$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \to 4!$$

$$\frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{10!}{4!}$$

Example 1.4

How many ways to choose 6 subjects without replacement if the order doesn't matter?

$$\underline{10}\ \underline{9}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5}$$

$$\left(\frac{10!}{4!}\right) \Big/ 6! = \binom{10}{6} \quad 10C6$$

$$6! \to (10-4)!$$

a b c d of 4 select 2

order does NOT matter w/out rep

ab ~~ba~~ ac ~~ca~~ bc ~~cb~~

ad ~~da~~ bd ~~db~~ cd ~~dc~~

$\boxed{\text{6 ways to choose}}$

$$\left(\frac{4!}{2!}\right) \Big/ 2! \qquad 2! \to (4-2)!$$

Permutations and Combinations

Permutations and Combinations

Definition: Permutations

Permutations are the number of ways to **arrange in order** r distinct objects when there are n total.

$$nPr = \frac{n!}{(n-r)!}$$

$10!$

$(10-6)!$

$\frac{10!}{4!}$

ex 1.3

Definition: Combinations

Combinations are the number of ways to choose (**order doesn't matter**) r objects from n without replacement.

$$nCr = \text{"n choose r"} = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$\frac{10}{(10-6)!} / 6!$

in ex 4

$n = 10$

$r = 6$

$n - r = 4$

Some combinations properties

- $\binom{n}{r} = \binom{n}{n-r}$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

$$\binom{n}{n-r} = \frac{n!}{(n-r)!\,(\cancel{n}+(\cancel{-n}+r))!} = \frac{n!}{(n-r)!\,r!}$$

"r"

- $\binom{n}{1} = n$

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = \frac{n\cdot\overbrace{(n-1)\cdot(n-2)\cdots(1)}^{\cancel{(n-1)!}}}{1!\,\underbrace{(n-1)\cdot(n-2)\cdots(1)}_{\cancel{(n-1)!}}} = \frac{n}{1} = n$$

- $\binom{n}{0} = 1$

$$\binom{n}{0} = \frac{n!}{0!\,(n-0)!} = \frac{n!}{1\times n!} = 1$$

$0! \downarrow = 1$

More Examples: order matters vs. not

More examples: order matters vs. not (1/2)

Example 2

Suppose we draw 2 cards from a standard deck without replacement. What is the probability that both are spades when

1. order matters?
2. order doesn't matter?

52 cards

13 faces/#s per suit

4 suits

★ Check if r! always cancels out for #2

$_{52}P_2$

(1) Let A = both cards spades

S = picking 2 cards

$52 \cdot 51 = \frac{52!}{(52-2)!}$

$\underline{52} \quad \underline{51}$

$|S| = 52 \cdot 51$

A

$\underline{13} \cdot \underline{12} \rightarrow |A| = 13 \cdot 12 \rightarrow {}_{13}P_2$

order matters

★ $P(\text{both spades}) = \frac{|A|}{|S|} = \frac{13 \cdot 12}{52 \cdot 51} = \left(\frac{13}{52}\right)\left(\frac{12}{51}\right)$

(2) A: $\underline{13} \cdot \underline{12}$ $|A| = \binom{13}{2} = {}_{13}C_2$

$|S| = \binom{52}{2} = {}_{52}C_2$

$\underline{52} \quad \underline{51}$

ORDER DOES NOT MATTER

★ $P(\text{both spades}) = \frac{\binom{13}{2}}{\binom{52}{2}} = \frac{\left(\frac{13!}{2!(11!)}\right)}{\left(\frac{52!}{2!(50!)}\right)}$

Table of different cases

See table on pg. 277 of textbook

- n = total number of objects
- r = number objects needed

Sample space vs event
n_s n_e
r r

	with replacement	without replacement
order matters	n^r	$nPr = \dfrac{n!}{(n-r)!}$
order doesn't matter	$\dbinom{n+r-1}{r}$	$nCr = \dbinom{n}{r} = \dfrac{n!}{r!(n-r)!}$