Calculus Review

Meike Niederhausen and Nicky Wakim

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Learning Objectives

1. Find derivatives of continuous functions with one variable
2. Find antiderivatives and integrals of functions with one variable

Differentiation

Find the *derivative* of the following function

Slope of fn $\quad \frac{d}{dx}(f(x)) \quad f'(x)$

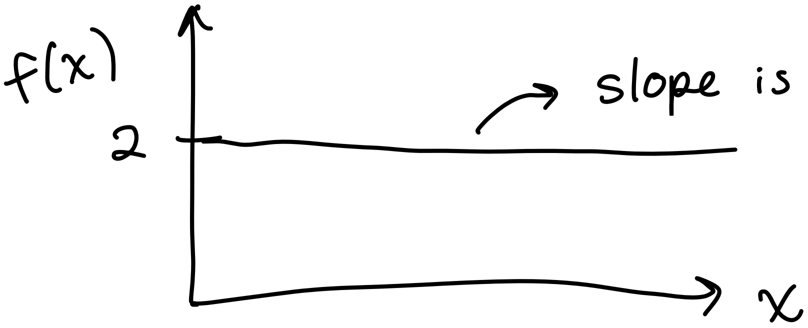
prime

Example 1.1

$f(x) = 2$

Derivative of a constant

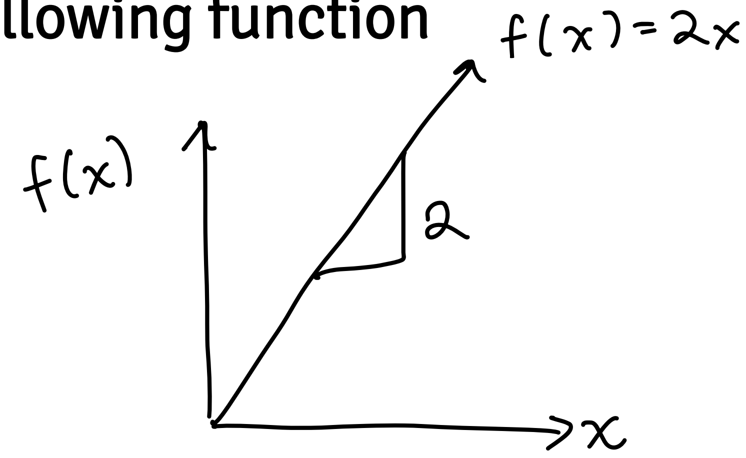
$$\frac{d}{dx}c = 0$$



slope is flat = 0

$\frac{d}{dx}(2)$ OR $f'(x) = 0$

Find the *derivative* of the following function



Example 1.2

$f(x) = 2x$

$$f'(x) = 2$$

$$\frac{d}{dx}(2x) = 2$$

Find the *derivative* of the following function

Example 1.3

$f(x) = \underline{2x} + \underline{2}$

$f(x) = 2x + 2$

$g(x) = 2x$

$f'(x) = g'(x)$

$$f'(x) = \frac{d}{dx}(2x + 2)$$

$$= \frac{d}{dx}(2x) + \frac{d}{dx}(2)$$

$$= 2 + 0$$

$$f'(x) = 2$$

Find the *derivative* of the following function

Example 1.4

$f(x) = x^2$

Derivative of x to a constant

$$\frac{d}{dx} x^n = nx^{n-1}$$

$$f'(x) = \underset{n}{2}\, x^{\overbrace{2-1}^{=1}} = 2x$$

Find the *derivative* of the following function

Example 1.5

$$f(x) = 3\sqrt{x} + \frac{2}{x} + 5$$

$$f(x) = 3x^{1/2} + 2x^{-1} + 5$$

$$= 3\left[\frac{1}{2}x^{\frac{1}{2}-1}\right] + 2\left(-1x^{-1-1}\right) + 0$$

$$= \frac{3}{2}x^{-1/2} - 2x^{-2}$$

$$f'(x) = \frac{3}{2\sqrt{x}} - \frac{2}{x^2}$$

Find the *derivative* of the following function

Example 1.6

$f(x) = e^x$

Derivative of exponential function

$$\frac{d}{dx}e^x = e^x$$

$f'(x) = e^x$

Find the *derivative* of the following function

Example 1.7

$f(x) = \underline{\ln(x)}$

Derivative of logarithm (natural)

$$\frac{d}{dx}\underline{\ln(x)} = \frac{1}{x}$$

$$f'(x) = \frac{1}{x}$$

ln or $\underline{\log}$ → natural log

$\log_e$

$\log_{10}$

Find the *derivative* of the following function

Example 1.8

$f(x) = x^2 e^x$

Product Rule

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

$$f(x) = \underbrace{x^2}_{h(x)} \underbrace{e^x}_{g(x)}$$

$$f'(x) = h'(x)g(x) + h(x)g'(x)$$

$$f'(x) = (2x)e^x + x^2 e^x$$

Find the *derivative* of the following function

Example 1.9

$f(x) = \frac{x^5}{2x+7}$ $\qquad x^5 \cdot \underbrace{(2x+7)^{-1}}$

Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x) \rightarrow \text{high}}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

low

$$= \frac{\text{low } d\text{'high} - \text{high } d\text{'low}}{\text{low} \cdot \text{low}}$$

$$f(x) = \frac{x^5 \rightarrow h(x): \text{high}}{2x+7 \rightarrow g(x): \text{low}}$$

$$f'(x) = \frac{(2x+7)(5x^4) - (x^5)(2)}{(2x+7)^2}$$

$$f'(x) = \frac{10x^5 + 35x^4 - 2x^5}{(2x+7)^2}$$

$$\boxed{f'(x) = \frac{8x^5 + 35x^4}{(2x+7)^2}}$$

Find the *derivative* of the following function

Example 1.10

$f(x) = e^{-2x+7}$

~~**Quotient Rule**~~

$$\cancel{\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}}$$

$f(x) = e^{-2x+7}$

$g(x) = -2x + 7$

$h(g(x)) = e^{g(x)}$

Chain Rule

$$\frac{d}{dx}\left(f\left(g(x)\right)\right)$$

$$= f'\left(g(x)\right) g'(x)$$

$$f'(x) = h'(g(x))\,g'(x)$$

$$= e^{g(x)}(-2 + 0)$$

$$f'(x) = -2e^{-2x+7}$$

Find the *derivative* of the following function

Example 1.11

$f(x) = \ln(x^2)$

Chain Rule

$$\frac{d}{dx} f\left(g(x)\right) = f'\left(g(x)\right) g'(x)$$

$f(x) = \ln(x^2)$

$g(x) = \underline{x^2}$

$h(g(x)) = \underline{\ln(g(x))}$

$$f'(x) = h'(g(x))\, g'(x)$$

$$= \frac{1}{g(x)}(2x)$$

$$= \frac{2x}{x^2}$$

$$\boxed{f'(x) = \frac{2}{x}}$$

Integration

Find the *antiderivative* of the following function

Example 2.1

$f(x) = 2$

deriving

$$\int_0^0 2\,dx = 2x + c$$

integrating

no bounds to int

$f(x) = 2x \quad g(x) = 2x + 2$

$g'(x) = f'(x) = 2$

$k(x) = 2 \rightarrow$ additional constant

check

$m(x) = 2x + c$

$m'(x) = 2 + 0$

Find the *antiderivative* of the following function

Example 2.2

$f(x) = x$

Integration of x to a constant

$$\int x^n dx = \frac{\boxed{x^{n+1}}}{\boxed{n+1}} + c$$

$$\int x^{1} dx = \frac{x^{1+1}}{1+1} + c$$

$n=1$

$$= \frac{x^2}{2} + c$$

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \cancel{(n+1)}\,\frac{x^{n+1-1}}{\cancel{n+1}} = x^n$$

1

Find the *antiderivative* of the following function

Example 2.3

$f(x) = \frac{1}{x}$

$$\int \frac{1}{x}\,dx = \ln|\underbrace{x}_{\text{x can be neg or pos.}}| + C$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln(-x)) = \frac{1}{(-x)}(-1) = \frac{1}{x}$$

Find the *antiderivative* of the following function

Example 2.4

$f(x) = x^{3/2}$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

$$\int x^{3/2} \, dx = \frac{x^{3/2+1}}{\left(\frac{3}{2}+1\right)} + c$$

$$= \frac{2x^{5/2}}{5} + c$$

Find the *antiderivative* of the following function

Example 2.5

$f(x) = e^x$

$$\frac{d}{dx} e^x = e^x$$

$$\int e^x dx = e^x$$

$$\int e^x \, dx = e^x + c$$

Find the *antiderivative* of the following function

Example 2.6

$f(x) = e^{-x}$

$$\int e^{-x}\,dx = (-1)e^{-x} + C$$

$$= -e^{-x} + C$$

$\underline{(1)}\, e^{g(x)}$

$g(x) = \underline{-x}$

$g'(x) = -1$

<u>check:</u>

$$\frac{d}{dx}\left(-e^{-x}\right)$$

$$= \left(-e^{-x}\right)(-1)$$

$$= e^{-x}$$

Find the *antiderivative* of the following function

Example 2.7

$f(x) = e^{-2x}$

$$\int \underline{e^{-2x}}\, dx = \underline{\frac{-1}{2}}\ \underline{e^{-2x}} + c$$

$$1$$

$$-2 \times w\cancel{hat} = 1$$

$$w = \frac{1}{-2} = \frac{-1}{2}$$

Solve the following integral

Example 3.1

$$\int_0^1 (2x + x^5)dx$$

$$\int_0^1 (2x + x^5)\, dx$$

$$= x^2 + \frac{1}{6}x^6 \Big|_0^1$$

$$= \left[(1)^2 + \frac{1}{6}(1)^6\right]_{x=1} - \left[0^2 + \frac{1}{6}0^6\right]_{x=0}$$

$$= 1 + \frac{1}{6} - 0 - 0$$

$$= \boxed{\frac{7}{6}}$$

Solve the following integral

Example 3.2

$$\int_2^3 e^{-x}\,dx$$

$g(x)$

U-substitution

$$\int f\big(g(x)\big)g'(x)dx = \int f(u)du$$

$u = -x$

$\frac{du}{dx} = \frac{d}{dx}(u) = -1$

$du = -1\,dx$

$dx = -1\,du$

$$\int_2^3 e^{-x}dx = \int_{u=-2}^{u=-3} e^u(-1\,du)$$

$$= -\int_{-2}^{-3} e^u\,du = -\left[\ e^u \Big|_{-2}^{-3}\ \right]$$

$$= -\left[e^{-3} - e^{-2}\right] = \boxed{e^{-2} - e^{-3}}$$

Solve the following integral

bounds: $x = 2 \rightarrow u = x^2 = 4$

$x = 3 \rightarrow u = x^2 = 3^2 = 9$

Example 3.3

$$\int_2^3 x e^{x^2} dx$$

$$u = x^2$$

$$\frac{d}{dx} u = 2x \rightarrow x\,dx = \frac{1}{2} du$$

$$\frac{du}{dx}$$

★ why can we treat as fraction

$$\int_2^3 x e^{x^2} dx = \int_4^9 \frac{1}{2} e^u du$$

$$= \frac{1}{2} \int_4^9 e^u du = \frac{1}{2} \left[e^u \Big|_4^9 \right]$$

$$= \boxed{\frac{1}{2}(e^9 - e^4)}$$

Solve the following integral

Example 3.4

$$\int_0^\infty x e^{-x}\,dx$$

Integrating by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

OR

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$u = x$

$\frac{du}{dx} = 1$

$du = 1dx$

$dv = e^{-x}dx$

$v = \int e^{-x}dx = -e^{-x}$

$$\int_0^\infty x e^{-x} dx = -xe^{-x}\Big|_0^\infty + \int_0^\infty (+e^{-x})dx$$

$$= \left[\underbrace{-\infty}_{\infty}\underbrace{e^{-\infty}}_{0} - (-0e^{-0})\right] + (-e^{-x})\Big|_0^\infty$$

$$= 0 - 0 + \left[\underbrace{-e^{-\infty}}_{=0} + (+e^{-0})\right]$$

$$= \boxed{1}$$

$\lim_{x\to\infty} e^{-x} \to 0$ "faster" than $\lim_{x\to\infty} x \to \infty$

Solve the following integral

Example 3.5

$$\int_1^2 x^2 \ln(x)dx$$

Solve the following integral

Example 3.6

$$\int_1^2 \ln(x)dx$$

Solve the following integral

Example 3.7

$$\int_1^2 x^2 e^x \, dx$$

e^x

$f(x) = x^2$
$f'(x) = 2x$
$f''(x) = 2$

$u = x^2$ $\quad dv = e^x dx$

$\frac{du}{dx} = 2x$ $\quad v = \int e^x dx = e^x$

$du = 2x\,dx$

$$\int_1^2 x^2 e^x dx = \underline{x^2 e^x \Big|_1^2} - \underline{\int_1^2 2x e^x dx}$$

$$= 2^2 e^2 - 1^2 e^1 - 2e^2$$

$$= 4e^2 - e^1 - 2e^2$$

$$= \boxed{2e^2 - e}$$

I by P #2:

$u = 2x$ $\quad dv = e^x dx$

$\frac{du}{dx} = 2$ $\quad v = e^x$

$$= 2x e^x \Big|_1^2 - \int_1^2 2e^x dx$$

$$= 4e^2 - 2e^1 - 2e^x \Big|_1^2$$

$$= 2e^2$$

☆ philosophy on integrals in actually solving our problems