Chapter 24: Continuous r.v.'s and pdf's

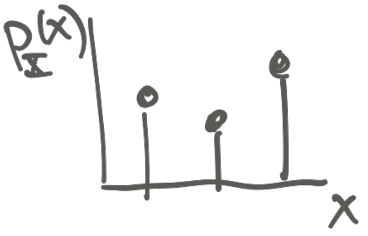
Recall from Chapter 7: Fig 24.1, p.301 Table

Discrete vs. Continuous r.v.'s

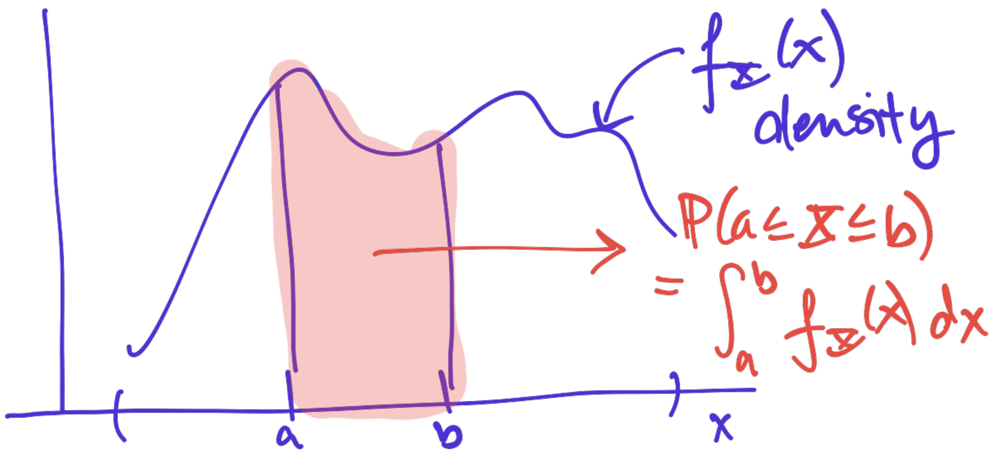
- For a **discrete** r.v., the set of possible values is either finite or can be put into a countably infinite list.
- **Continuous** r.v.'s take on values from continuous *intervals*, or unions of continuous intervals.

How to define probabilities for continuous r.v.'s?

Discrete r.v. X : pmf $p_X(x) = \mathbb{P}(X=x)$



Continuous r.v. X



Definition 24.1 (Probability density function).
The probability distribution, or ***probability density function (pdf)****, of a continuous random variable X is a function $f_X(x)$, such that for all real values a, b with $a \le b$,*

$$\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)dx$$

Remarks:

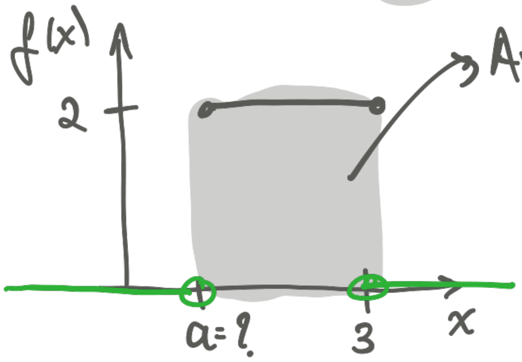
(1) Note that $f_X(x) \neq \mathbb{P}(X = x)$!!!

(2) In order for $f_X(x)$ to be a pdf, it needs to satisfy the properties
- $f_X(x) \geq 0$ for all x
- $\int_{-\infty}^{\infty} f_X(x)dx = 1$

Example 24.2. *Let $f_X(x) = 2$, for $a \le x \le 3$.*

$f_X(x) = \begin{cases} 2 & 2.5 \le x \le 3 \\ 0 & \text{else} \end{cases}$

(1) *Find the value of a so that $f_X(x)$ is a pdf.*



Area = 1

$2 \cdot (3-a) = 1$

$\underline{\underline{a = 2.5}}$

OR $\int_a^3 2\,dx = 1$

(2) *Find $\mathbb{P}(2.7 \le X \le 2.9)$.*

$\mathbb{P}(2.7 \le X \le 2.9) = \int_{2.7}^{2.9} 2\,dx = 2x\Big|_{2.7}^{2.9} = 2(2.9 - 2.7)$

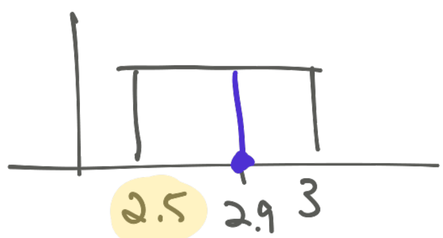
$= 2(.2) = \underline{\underline{.4}}$

$\mathbb{P}(a \le X \le b) = \mathbb{P}(a < X < b)$

(3) *Find $\mathbb{P}(2.7 < X \le 2.9)$.*

$\mathbb{P}(2.7 < X \le 2.9) = \int_{2.7}^{2.9} 2\,dx = \ldots = \underline{\underline{0.4}}$

(4) *Find $\mathbb{P}(X = 2.9)$.*



$\mathbb{P}(X = a) = 0$

$\mathbb{P}(X = 2.9) =$

$= \int_{2.9}^{2.9} 2\,dx = 0$

(5) *Find $\mathbb{P}(X \le 2.8)$.*

$\mathbb{P}(X \le 2.8) = \int_{\boxed{2.5}}^{2.8} 2\,dx = 2x\Big|_{2.5}^{2.8} = 2(2.8 - 2.5) = .6$

$F_X(x)$

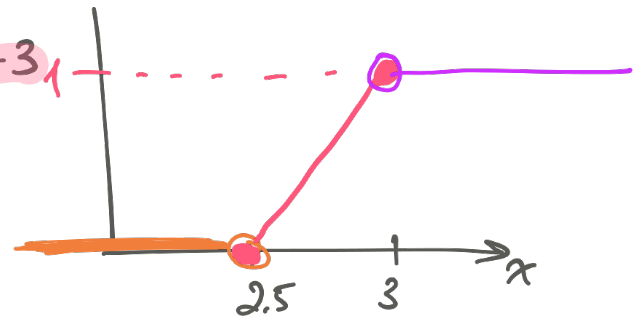
Definition 24.3 (Cumulative distribution function).
The ***cumulative distribution function (cdf)*** *of a continuous random variable X, is the function $F_X(x)$, such that for all real values of x,*

$$F_X(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(s)ds$$

Example 24.4. *Let $\boxed{f_X(x) = 2}$, for $\boxed{2.5} \le x \le 3$. Find $F_X(x)$.*

$$F_X(x) = \mathbb{P}(X \le x) = \int_{2.5}^{x} 2\, ds \quad \text{dummy variable} \quad = 2s\Big|_{2.5}^{x} = 2(x-2.5)$$

$$= \begin{cases} 0 & \text{for } x < 2.5 \\ 2x-5 & \text{for } 2.5 \le x \le 3 \\ 1 & \text{for } x > 3 \end{cases}$$

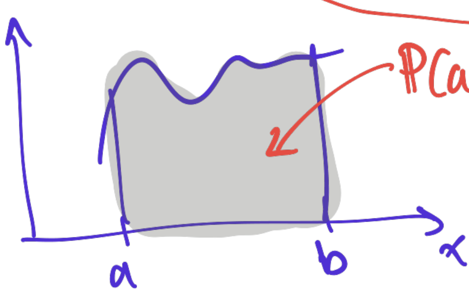


Remarks:
In general, $F_X(x)$ is increasing and

- $\lim_{x\to-\infty} F_X(x) = 0$
- $\lim_{x\to\infty} F_X(x) = 1$

$$\mathbb{P}(X > a) = 1 - \mathbb{P}(X \le a) = 1 - F_X(a)$$

$$\mathbb{P}(a \le X \le b) = F_X(b) - F_X(a)$$



$$\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)dx = F_X(b) - F_X(a)$$

FTC!!

Theorem 24.5.
If X is a continuous random variable with pdf $f_X(x)$ and cdf $F_X(x)$, then for all real values of x at which $F'_X(x)$ exists,

$$\frac{d}{dx}F_X(x) = F'_X(x) = f_X(x)$$

Example 24.6. *Let X be a r.v. with cdf*

$$F_X(x) = \begin{cases} 0 & x < 2.5 \\ 2x - 5 & 2.5 \leq x \leq 3 \\ 1 & x > 3 \end{cases}$$

Find the pdf $f_X(x)$.

Solution:

$$f(x) = F'(x) = \begin{cases} 0 & x < 2.5 \\ 2 & 2.5 \leq x \leq 3 \\ 0 & x > 3 \end{cases} = 2 \text{ for } 2.5 \leq x \leq 3 \quad \checkmark$$

Example 24.7. *Let X be a r.v. with pdf $f_X(x) = 2e^{-2x}$, for $x > 0$.*

Exponential

(1) *Show $f_X(x)$ is a pdf.*

① $f_X(x) \geq 0$, all x : $2e^{-2x} \geq 0$ ✓

② $\int_{-\infty}^{\infty} f(x)\,dx = 1$ $\quad \int_0^{\infty} 2e^{-2x}\,dx$ $\quad u = -2x \quad du = -2dx \quad \frac{du}{dx} = -2$

$$= \int_0^{-\infty} \frac{2e^u}{-2}\,du = (-)(-)\int_{-\infty}^{0} e^u\,du = e^u \Big|_{-\infty}^{0} = e^0 - \lim_{u \to -\infty} e^u = 1 - 0 = 1 \checkmark$$

"$e^{-\infty}$"

(2) *Find $\mathbb{P}(1 \leq X \leq 3)$.*

$$\mathbb{P}(1 \leq X \leq 3) = \int_1^3 2e^{-2x}\,dx = -e^{-2x}\Big|_1^3 - (e^{-6} - e^{-2}) = e^{-2} - e^{-6}$$

$(-)\,-2e^{-2x}$

$f_X(x) = 2e^{-2x}, \text{for } x > 0$

(3) *Find* $F_X(x)$.

$$F_X(x) = \mathbb{P}(X \le x) = \int_0^x 2e^{-2s}\,ds = -e^{-2s}\Big|_0^x = -e^{-2x} - (-e^0)$$

$$= \begin{cases} 0 & x \le 0 \\ 1 - e^{-2x} & x > 0 \end{cases}$$

Where is $F(x) = 1$? $\lim_{x \to \infty} 1 - e^{-2x} = 1$, with $e^{-2x} \to 0$

(4) *Given* $F_X(x)$, *find* $f_X(x)$.

$$f_X(x) = \frac{d}{dx} F_X(x) = F_X'(x) = \frac{d}{dx}\left(1 - e^{-2x}\right) = 2e^{-2x}, \quad \text{for } x > 0 \quad ✓$$

(5) *Find* $\mathbb{P}(X \ge 1 | X \le 3)$.

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \text{ and } B)}{\mathbb{P}(B)}$$

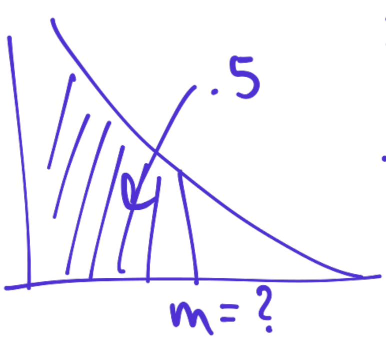
$$\mathbb{P}(X \ge 1 | X \le 3) = \frac{\mathbb{P}(1 \le X \le 3)}{\mathbb{P}(X \le 3)}$$

$$\frac{\int_1^3 2e^{-2x}\,dx}{\int_0^3 2e^{-2x}\,dx}$$

$F_X(x) = 1 - e^{-2x}$

$$= \frac{F_X(3) - F_X(1)}{F_X(3)} = \frac{(\cancel{1} - e^{-6}) + (-\cancel{1} + e^{-2})}{1 - e^{-6}} = \boxed{\frac{e^{-2} - e^{-6}}{1 - e^{-6}}}$$

(6) *Find the median of the distribution of* X.



Find m s.th.

$$.5 = \mathbb{P}(X \le m) = \int_0^m 2e^{-2x}\,dx = \ldots \overset{\text{(OR)}}{=} F(m) = 1 - e^{-2m}$$

$$.5 = 1 - e^{-2m}$$
$$.5 = e^{-2m}$$

$$\ln(.5) = -2m \ln(e)$$
$$m = \frac{\ln(.5)}{-2} = 0.34657$$