Chapter 25: Joint densities

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Learning Objectives

1. Solve double integrals in our mini lesson!
2. Calculate probabilities for a pair of continuous random variables
3. Calculate a *joint and marginal* probability density function (pdf)
4. Calculate a *joint and marginal* cumulative distribution function (CDF) from a pdf

Double Integrals Mini Lesson (1/3)

Mini Lesson Example 1

Solve the following integral:

$$\int_{2}^{3}\int_{0}^{1} \mathrm{xydydx}$$

Double Integrals Mini Lesson (2/3)

Mini Lesson Example 2

Solve the following integral:

$$\int_{2}^{3}\int_{0}^{1}(x + y)dydx$$

Double Integrals Mini Lesson (3/3)

Do this problem at home for extra practice. The solution is available in Meike's video!

Mini Lesson Example 3

Solve the following integral:

$$\int_2^3 \int_0^1 e^{x+y} dydx$$

How to define the joint pdf for continuous RVs?

For a single continuous RV X is a function $f_X(x)$, such that for all real values a, b with $a \le b$,

$$\mathbb{P}(a \le X \le b) = \int_a^b f_X(x)dx$$

For two continuous RVs (X and Y), we can define the **joint pdf**, $f_{X,Y}(x, y)$, such that for all real values a, b, c, d with $a \le b$ and $c \le d$,

$$\mathbb{P}(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f_{X,Y}(x, y)dydx$$

Important properties of the joint pdf

1. Note that $f_{X,Y}(x, y) \neq \mathbb{P}(X = x, Y = y)$!!!
2. In order for $f_{X,Y}(x, y)$ to be a pdf, it needs to satisfy the properties
   - $f_{X,Y}(x, y) \geq 0$ for all x, y
   - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dxdy = 1$

What is the joint cumulative distribution function?

Definition: Joint cumulative distribution function (Joint CDF)

The **joint cumulative distribution function (cdf)** of continuous random variables X and Y, is the function $F_{X,Y}(x, y)$, such that for all real values of x and y,

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s, t)dtds$$

Remarks:

- The definition above for $F_{X,Y}(x, y)$ is a **function** of x and y.
- The joint cdf at the point (a, b), is

$$F_{X,Y}(a, b) = \mathbb{P}(X \leq a, Y \leq b) = \int_{-\infty}^{a} \int_{-\infty}^{b} f_{X,Y}(s, t)dtds$$

What are the marginal pdf's?

Definition: Marginal pdf's

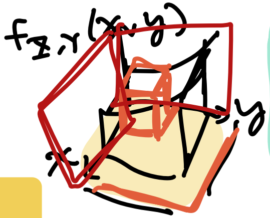
Suppose X and Y are continuous r.v.'s, with joint pdf $f_{X,Y}(x, y)$. Then the **marginal probability density functions** are

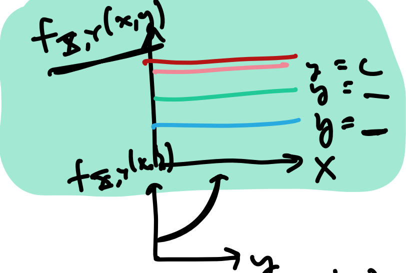
int over all y

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)dx$$

Example of joint pdf





Example 1.1

Let $f_{X,Y}(x,y) = \frac{3}{2}y^2$, for $0 \leq x \leq 2,\ 0 \leq y \leq 1.$

1. Find $\mathbb{P}(0 \leq X \leq 1, 0 \leq Y \leq \frac{1}{2})$.

$$P(0 \leq X \leq 1,\ 0 \leq Y \leq \tfrac{1}{2})$$

$$= \int_0^1 \int_0^{1/2} \frac{3}{2} y^2 \, dy\, dx$$

$$= \int_0^1 \left[\frac{1}{2} y^3 \right]_{y=0}^{y=1/2} dx$$

$$= \int_0^1 \left[\frac{1}{2}\left(\frac{1}{2}\right)^3 - \frac{1}{2}(0)^3 \right] dx$$

$$= \int_0^1 \frac{1}{16}\, dx = \left[\frac{1}{16} x \right]_{x=0}^{x=1}$$

$$= \frac{1}{16}(1) - \frac{1}{16}(0) = \frac{1}{16}$$

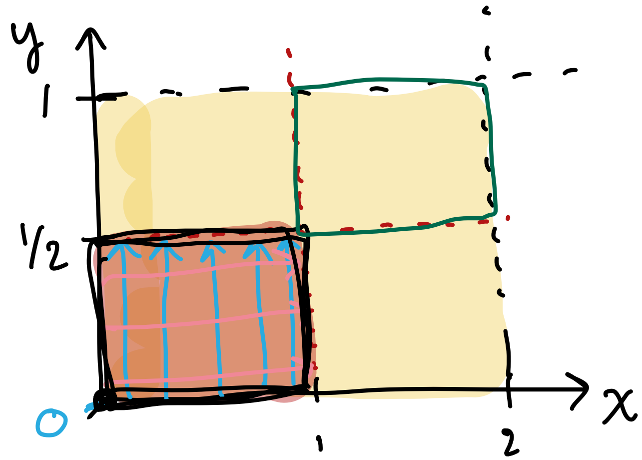
$$P(0 \leq X \leq 1,\ 0 \leq Y \leq 1/2) = 1/16$$

Steps in problem

1. set up domain of pdf
2. shade in area of probability of interest
3. set up integral: dydx or dxdy?

domain of y and domain of x don't depend on each other & pdf is only in terms of y

① & ②



$$f_X(x) = \exp\left(\frac{\quad}{\quad} \right)$$

Example of joint pdf

$f_X(x)$: int out y $f_Y(y)$: int out x

Example 1.2

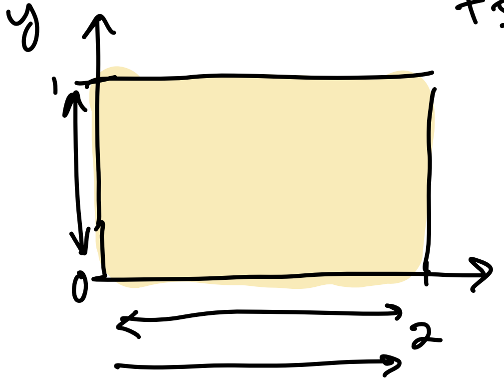
Let $f_{X,Y}(x,y) = \frac{3}{2}y^2$, for $0 \le x \le 2,\ 0 \le y \le 1$.

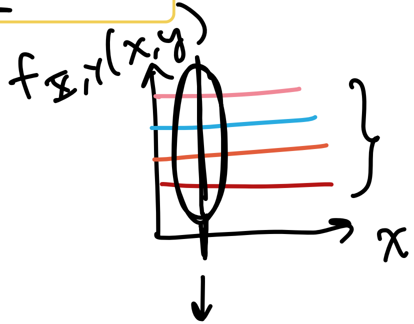
2. Find $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_0^1 \frac{3}{2}y^2\,dy = \frac{1}{2}y^3 \Big|_{y=0}^{y=1}$$

$$= \frac{1}{2}(1)^3 - \frac{1}{2}(0)^3$$

$$= \frac{1}{2}$$





$$f_X(x) = \frac{1}{2} \quad \text{for } 0 \le x \le 2$$

$$f_Y(y) = \int_0^2 \frac{3}{2}y^2\,dx = \frac{3}{2}y^2 x \Big|_{x=0}^{x=2}$$

$$= \frac{3}{2}y^2(2) - \frac{3}{2}y^2(0)$$

$$f_Y(y) = 3y^2 \quad \text{for } 0 \le y \le 1$$

CDF

$P(y_1 \le Y \le y_2)$ → $P(Y \le y) = \int_{-\infty}^{y} 3t^2\,dt$ → $F_Y(y)$

Example of a *more complicated* joint pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 2.1

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

1. Find $f_X(x)$ and $f_Y(y)$.

Example of a *more complicated* joint pdf

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 2.2

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \leq x \leq y$.

2. Find $\mathbb{P}(Y < 3)$.

Let's complicate this even more!

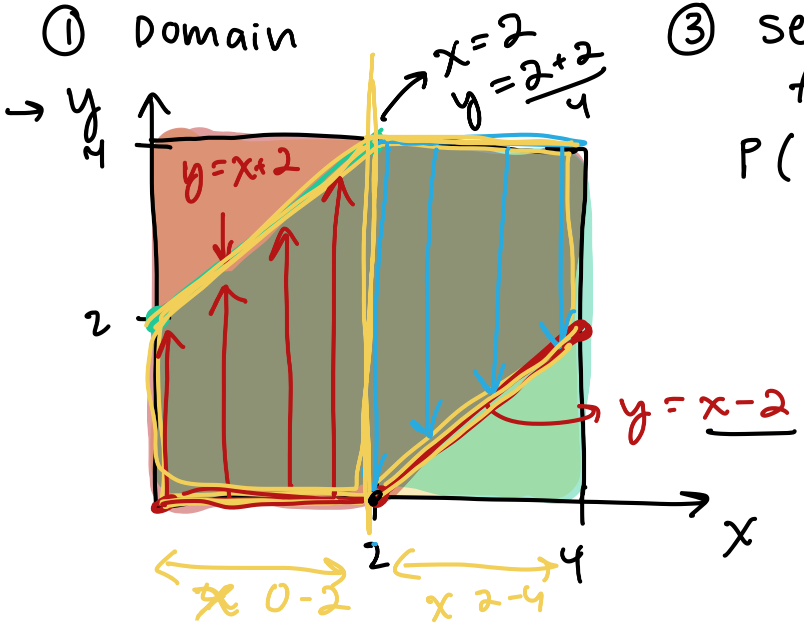
$f_{X,Y}(x,y) = \frac{1}{16}$ for $0 \le X \le 4,\ 0 \le Y \le 4$

Example 3.1

Let X and Y have constant density on the square $0 \le X \le 4, 0 \le Y \le 4$.

1. Find $\mathbb{P}(|X - Y| < 2)$

① Domain



② shade in prob $P(|X-Y|<2)$

$-2 < x-y < 2$

$-2 < x - y$, $+y+2$, $+2+y$: $y < x+2$

$x - y < 2$, $-2+y$, $+y-2$: $x-2 < y$

$4 \times 4 \cdot b = 1$

$\int_0^4\int_0^4 b\, dy\, dx = 1$

$4^2 b = 1$, $b = \frac{1}{16}$

③ set up integral for probability

$$P(|X-Y|<2) = \int_0^2\int_0^{x+2} \frac{1}{16}\, dy\, dx + \int_2^4\int_{x-2}^{4} \frac{1}{16}\, dy\, dx$$

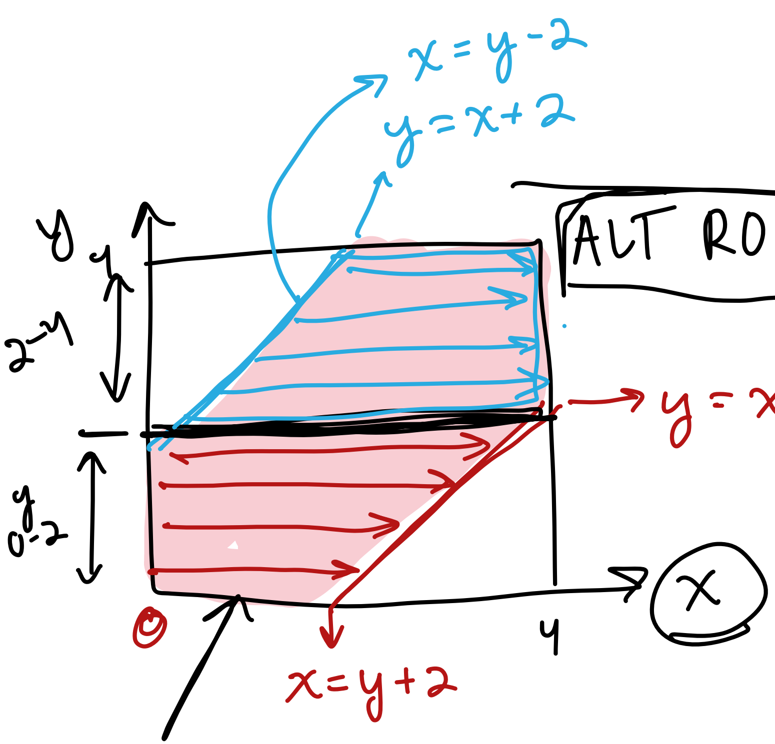
$$= \int_0^2 \left[\frac{1}{16} y\right]_{y=0}^{y=x+2} dx + \int_2^4 \left[\frac{1}{16} y\right]_{y=x-2}^{y=4} dx$$

$$= \int_0^2 \frac{1}{16}(x+2)\, dx + \int_2^4 \left[\frac{1}{16}4 - \frac{1}{16}(x-2)\right] dx$$

$$= \int_0^2 \frac{1}{16}(x+2)\,dx + \int_2^4 \left(\frac{1}{4} - \frac{1}{16}(x-2)\right)dx$$

$$= \left[\frac{1}{32}x^2 + \frac{1}{8}x\right]_{x=0}^{x=2} + \left[\frac{1}{4}x - \frac{1}{32}x^2 - \frac{1}{8}x\right]_{x=2}^{x=4}$$

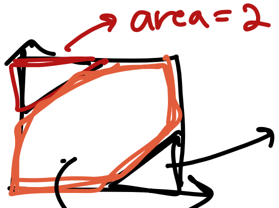
solve this @ home



ALT ROUTE

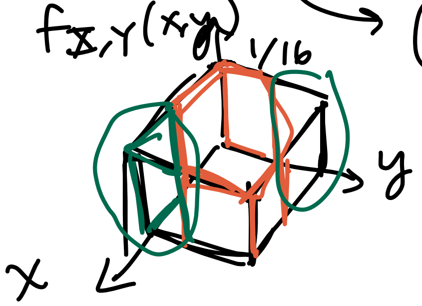
$$\int_0^2 \int_0^{y+2} \frac{1}{16}\,dx\,dy + \int_2^4 \int_{y-2}^{4} \frac{1}{16}\,dx\,dy$$

ALT ALT ROUTE



$\frac{1}{2}(2)(2) = 2$

$(4 \cdot 4) - 2 - 2 = 12$



$P(|X - Y| < 2)$

$= \frac{12}{16} = 3/4$

Let's complicate this even more!

$Z = X + Y$ $\quad$ $1 + 2$, $2 + 1$

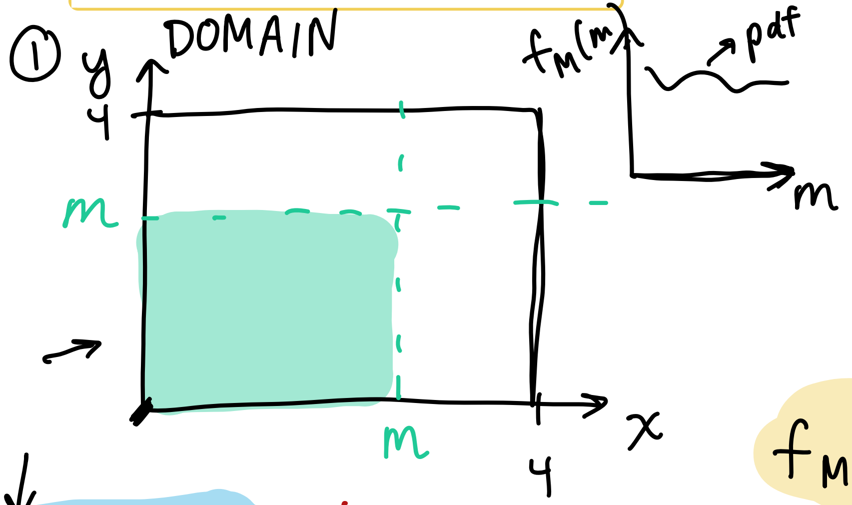
- M is a transformation of X & Y
- CDF method: start by finding the CDF (which includes probabilities) & then find pdf from CDF: $F_M(m) = \int f_M(m)\,dm \rightarrow f_M(m) = \frac{d}{dm} F_M(m)$

Example 3.1

Let X and Y have constant density on the square $0 \le X \le 4, 0 \le Y \le 4$.

2. Let $M = \max(X, Y)$. Find the pdf for M, that is $f_M(m)$.

① DOMAIN



$$F_M(m) = P(M \le m) = P(\max(X,Y) \le m)$$
$$= P(X \le m, Y \le m)$$
$$= \int_0^m \int_0^m \frac{1}{16}\, dy\, dx = \int_0^m \left[\frac{y}{16}\right]_{y=0}^{y=m} dx$$
$$= \int_0^m \frac{m}{16}\, dx = \frac{m}{16} x \Big|_{x=0}^{x=m} = \frac{m^2}{16}$$

($\frac{m}{16}$: constant)

$$f_M(m) = \frac{d}{dm} F_M(m) = \frac{2m}{16} = \frac{1}{8} m \quad \text{for } 0 \le m \le 4$$

$$f_{X,Y}(x,y) = \frac{1}{16} \text{ for } 0 \le x \le 4, 0 \le y \le 4$$

joint pdf of X & Y → (transformation) → CDF of M → (derivative) → pdf of M

dimension from 2 to 1 in domain

translate domain of X & Y into M

Let's complicate this even more!

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 3.3

Let X and Y have constant density on the square $0 \leq X \leq 4, 0 \leq Y \leq 4$.

3. Let $Z = \min(X, Y)$. Find the pdf for Z, that is $f_Z(z)$.

Let’s complicate this even further!

Example 4

Let X and Y have joint density $f_{X,Y}(x, y) = \frac{8}{5}(x + y)$ in the region $0 < x < 1,\ \frac{1}{2} < y < 1$. Find the pdf of the r.v. Z, where $Z = XY$.