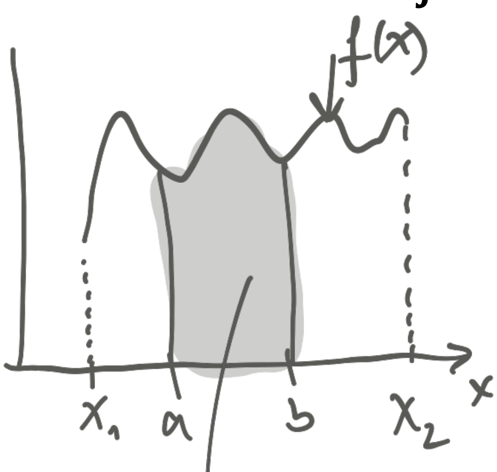
Chapter 25: Joint densities

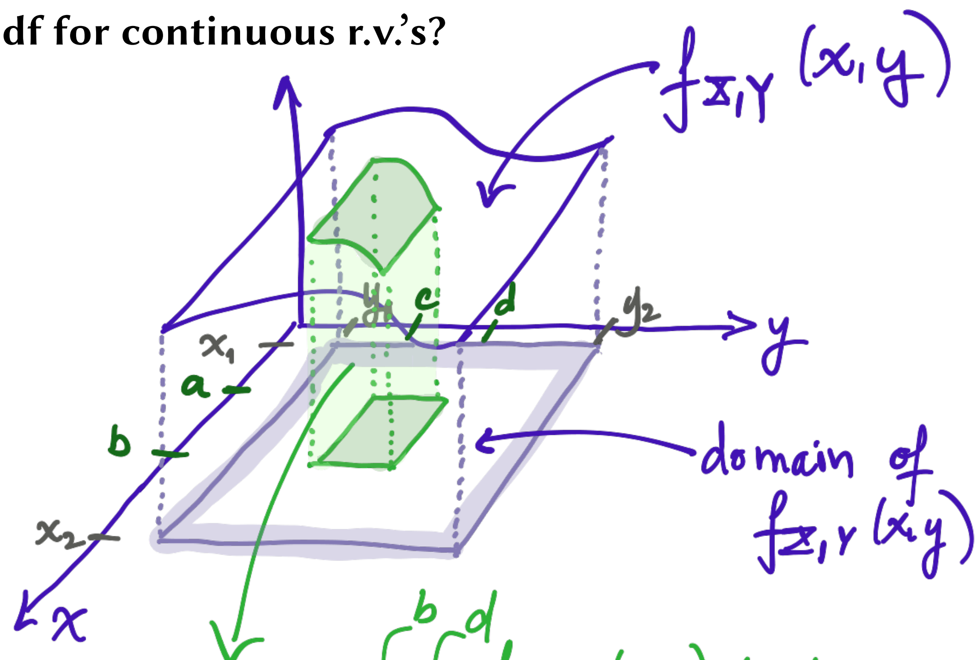
Recall from Chapter 24, that the probability distribution, or **probability density function (pdf)**, of a continuous random variable X is a function $f_X(x)$, such that for all real values a, b with $a \leq b$,

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f_X(x)dx.$$

How to define the joint pdf for continuous r.v.'s?



$\mathbb{P}(a \leq X \leq b)$



$$\text{Volume} = \int_a^b \int_c^d f_{X,Y}(x,y)\,dy\,dx$$

$$= \mathbb{P}(a \leq X \leq b,\ c \leq Y \leq d)$$

Remarks:

(1) Note that $f_{X,Y}(x,y) \neq \underbrace{\mathbb{P}(X = x, Y = y)}_{=0}$!!!

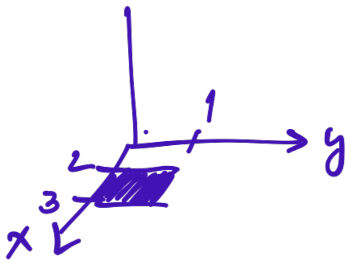
(2) In order for $f_{X,Y}(x,y)$ to be a pdf, it needs to satisfy the properties

- $f_{X,Y}(x,y) \geq 0$ for all x, y
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)dxdy = 1$

Double Integrals Mini Lesson

Example 25.1. *Solve the following integrals.*

(1) $\int_2^3\left(\int_0^1 xydy\right)dx$



$$= \int_2^3\left(x\int_0^1 y\,dy\right)dx = \int_2^3\left(x\,\frac{y^2}{2}\Big|_{0}^{1}\right)dx$$

y=

y=0

$$= \int_2^3 x\left(\frac{1}{2}-0\right)dx = \int_2^3 \frac{x}{2}\,dx$$

$$= \frac{x^2}{4}\Big|_2^3 = \frac{1}{4}(9-4) = \boxed{\frac{5}{4}}$$

(2) $\int_2^3\int_0^1 (x+y)dydx$

$$= \int_2^3\int_0^1 (x+y)\,dy\,dx = \int_2^3 \left(xy+\frac{y^2}{2}\right)\Big|_{y=0}^{y=1} dx$$

$$= \int_2^3 \left(x+\frac{1}{2}-0\right)dx = \frac{x^2}{2}+\frac{x}{2}\Big|_2^3 = \frac{9}{2}+\frac{3}{2}-\left(\frac{4}{2}+\frac{2}{2}\right)$$

$$= \boxed{3}$$

(3) $\int_2^3\int_0^1 e^{x+y}dydx$

$$\int_2^3\int_0^1 e^x e^y\,dy\,dx = \int_2^3 e^x e^y\Big|_{y=0}^{y=1} dx = \int_2^3 e^x\left(e^1-e^0\right)dx$$

$$= \int_2^3 (e-1)e^x\,dx = (e-1)e^x\Big|_2^3 = \boxed{(e-1)(e^3-e^2)}$$

Definition 25.2 (Joint cumulative distribution function).
The ***joint cumulative distribution function (cdf)*** *of continuous random variables X and Y, is the function $F_{X,Y}(x, y)$, such that for all real values of x and y,*

$$F_{X,Y}(x, y) = \mathbb{P}(X \leq x, Y \leq y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\,dt\,ds$$

Remarks:

- The definition above for $F_{X,Y}(x, y)$ is a **function** of x and y.
- The joint cdf at the point (a, b), is

$$F_{X,Y}(a, b) = \mathbb{P}(X \leq a, Y \leq b) = \int_{-\infty}^{a} \int_{-\infty}^{b} \underbrace{f_{X,Y}(s,t)\,dt\,ds}$$

OR: $\int f_{X,Y}(x,y)\,dy\,dx$

Definition 25.3 (Marginal pdf's).
Suppose X and Y are continuous r.v.'s, with joint pdf $f_{X,Y}(x, y)$. Then the ***marginal probability density functions*** *are*

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dy$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dx$$

$$p_X(x) = \sum_{\{\text{all } y\}} p_{X,Y}(x, y)$$