Chapter 26: Independent Continuous RVs

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Learning Objectives

1. Show that a joint pdf consists of two independent, continuous RVs.
2. Combine two independent RVs into one joint pdf or CDF.

How do we represent independent continuous RVs in a joint pdf?

What do we know about independence for events and discrete RVs?

For events: If $A \perp B$

$$P(A \cap B) = P(A)P(B)$$

$$P(A|B) = P(A)$$

For discrete RVs: If $X \perp Y$

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

$$F_{X,Y}(x, y) = F_X(x)F_Y(y)$$

$$\longrightarrow p_{X|Y}(x|y) = p_X(x)$$

$$p_{Y|X}(y|x) = p_Y(y)$$

What does it mean for continuous r.v.'s to be independent?

For continuous RVs: If $X \perp Y$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)$$

$$f_{X|Y}(x|y) = f_X(x)$$

Constructing a joint pdf from two independent, continuous RVs

Example 1.1

Let X and Y be independent r.v.'s with $f_X(x) = \frac{1}{2}$, for $0 \le x \le 2$ and $f_Y(y) = 3y^2$, for $0 \le y \le 1$.

1. Find $f_{X,Y}(x, y)$.

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$= \left(\tfrac{1}{2}\right)\left(3y^2\right)$$

$$f_{X,Y}(x,y) = \frac{3}{2} y^2 \quad \text{for} \quad \begin{array}{l} 0 \le x \le 2 \\ 0 \le y \le 1 \end{array}$$

[ex of domain not ind

$$0 \le x \le y/2$$

Constructing a joint pdf from two independent, continuous RVs

Example 1.2

Let X and Y be independent r.v.'s with $f_X(x) = \frac{1}{2}$, for $0 \le x \le 2$ and $f_Y(y) = 3y^2$, for $0 \le y \le 1$.

2. Find $\mathbb{P}(0 \le X \le 1, 0 \le Y \le \frac{1}{2})$.

$$P(0 \le X \le 1, 0 \le Y \le \tfrac{1}{2})$$
$$= F_{X,Y}(x=1, y=1/2) - \underbrace{F_{X,Y}(x=0, y=0)}_{0}$$
$$= F_X(x=1)\, F_Y(y=\tfrac{1}{2}) \longrightarrow = P(X \le 1) \cdot P(Y \le \tfrac{1}{2})$$
$$= \int_{-\infty}^{1} f_X(x)\, dx$$
$$= \left[\int_0^1 \underbrace{\tfrac{1}{2}}_{f_X(x)} dx\right]\left[\int_0^{1/2} 3y^2\, dy\right]$$
$$= \left[\tfrac{1}{2}x \Big|_{x=0}^{x=1}\right]\left[y^3 \Big|_{y=0}^{y=1/2}\right]$$
$$= \left[\tfrac{1}{2}(1) - \tfrac{1}{2}(0)\right]\left[\left(\tfrac{1}{2}\right)^3 - (0)^3\right]$$
$$= \left(\tfrac{1}{2}\right)^4 = \frac{1}{16}$$

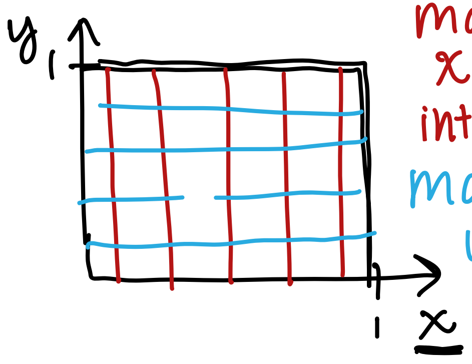
OR $P(0 \le X \le 1, 0 \le Y \le 1/2)$
$$= \int_0^1 \int_0^{1/2} \tfrac{3}{2} y^2\, dy\, dx$$
$$= \cdots$$
$$= 1/16$$

Showing independence from joint p~~m~~df

Example 2.1

Let $f_{X,Y}(x, y) = 18x^2y^5$ for $0 \le x \le 1,\ 0 \le y \le 1$.

1. Are X and Y independent?



marginal of x means int over all y

marginal of y means int over all x

What is $f_X(x)$ & $f_Y(y)$? And does $f_{X,Y}(x,y) = f_X(x)f_Y(y)$?

$$f_X(x) = \int_{y=0}^{y=1} f_{X,Y}(x,y)\,dy$$

$$= \int_{y=0}^{y=1} \underbrace{18x^2}_{\text{constant}} y^5\,dy = 3x^2y^6 \Big|_{y=0}^{y=1}$$

$$= 3x^2(1)^6 - 3x^2(0)^6$$

$$f_X(x) = 3x^2 \text{ for } 0 \le x \le 1$$

$$f_Y(y) = \int_{x=0}^{x=1} f_{X,Y}(x,y)\,dx = \int_0^1 18x^2y^5\,dx$$

$$f_Y(y) = 6y^5 \text{ for } 0 \le y \le 1$$

$$f_X(x)f_Y(y) = (3x^2)(6y^5) = 18x^2y^5$$

for $0 \le x \le 1$, $0 \le y \le 1$

$$\Rightarrow X \perp Y$$

pdf defined

$f_{X,Y}(x,y) = \_\_\_ \begin{cases} a \le x \le b \ \& \\ c \le y \le d \end{cases}$

$f_{X,Y}(x,y) = 0 \begin{cases} x < a \text{ or} \\ x > b \text{ or} \\ y < c \text{ or} \\ y > d \end{cases}$

$f_{X,Y}(x,y) = \_\_\_$

$f_{X,Y}(x=a_1, y=c_1) = \underline{18(a_1^2)(c_1^5)} \to P(X=a_1, Y=c_1) = 0$

CDF

$F_{X,Y}(x,y) \begin{cases} a \le x \le b \\ c \le y \le d \end{cases}$

$F_{X,Y}(x,y) = 0 \begin{cases} x < a \\ \text{OR } y < c \end{cases}$

$F_{X,Y}(x,y) = \cancel{F_Y(y)} \begin{cases} x > b \\ \& \ c \le y \le d \end{cases}$ for ind only

$= F_Y(y) \underbrace{F_X(x)}_{1} = F_Y(y)$

$$f_{X,Y}(x,y) \neq P(X = x, Y = y)$$

$a \le x \le b$
$c \le y \le d$

$$P(a_1 \le X \le a_2, \underline{c_1} \le Y \le c_2)$$

$$= \int_{c_1}^{c_2} \int_{a_1}^{a_2} f_{X,Y}(x,y)\, dx\, dy$$

$$= \underbrace{F_{X,Y}(X = a_2, Y = c_2)}_{= \int_{-\infty}^{c_2} \int_{-\infty}^{a_2} f_{X,Y}(x,y)\, dx\, dy} - F_{X,Y}(X = a_1, Y = c_1)$$

$-\infty \to c$
$-\infty \to a$

~~Showing independence from joint pmf~~ Finding CDF from two ind RVs

Example 2.2 $\int\int f_{X,Y}(x,y)\,dy\,dx$

Let $f_{X,Y}(x,y) = 18x^2y^5$, for $0 \le x \le 1,\ 0 \le y \le 1$.

2. Find $F_{X,Y}(x,y)$.

$$F_{X,Y}(x,y) = \underline{F_X(x)}\,\underline{F_Y(y)}$$

$f_X(x) = 3x^2$

① Find marginal CDFs.

$$F_X(x) = P(X \le x) = \int_0^x f_X(t)\,dt = \int_0^x 3t^2\,dt$$

$$= t^3\Big|_{t=0}^{t=x} = x^3 \text{ for } 0 \le x \le 1$$

$$F_Y(y) = P(Y \le y) = \int_0^y f_Y(s)\,ds = \ldots = y^6 \quad 0 \le y \le 1$$

② bring marginals together

$$F_{X,Y}(x,y) = \underbrace{F_X(x)}_{x^3}\,\underbrace{F_Y(y)}_{1} = \left(\begin{cases} 0 & x < 0 \\ x^3 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}\right)\left(\begin{cases} 0 & y < 0 \\ y^6 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}\right)$$

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 0 \text{ OR } y < 0 \\ x^3y^6 & 0 \le x \le 1 \ \& \ 0 \le y \le 1 \\ x^3 & 0 \le x \le 1 \quad y > 1 \\ y^6 & 0 \le y \le 1 \ \& \ x > 1 \\ 1 & x > 1 \ \& \ y > 1 \end{cases}$$

① given joint pdf
② get marg pdf
③ int marg pdf for marg CDF
④ mult marg CDF's

Example

Do this problem at home for extra practice. The solution is available in Meike's video!

Example 3

Let $f_{X,Y}(x, y) = 2e^{-(x+y)}$, for $0 \le x \le y$. Are X and Y independent?

Final statement on independence

1. If $f_{X,Y}(x, y) = g(x)h(y)$, where $g(x)$ and $h(y)$ are pdf's, then X and Y are independent.
   - The domain of the joint pdf needs to be independent as well!!

2. If $F_{X,Y}(x, y) = G(x)H(y)$, where $G(x)$ and $H(y)$ are cdf's, then X and Y are independent.
   - The domain of the joint CDF needs to be independent as well!!