Chapter 27: Conditional Distributions

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Learning Objectives

1. Calculate the conditional probability density from a joint pdf

Conditional probabilities we've seen before

What do we know about conditional probabilities for events and discrete RVs?

For events:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

For discrete RVs:

$$p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$$

What does it mean for conditional densities of continuous RVs?

For continuous RVs:

$$\frac{\text{joint pdf}}{\text{marg pdf}}$$

$$\rightarrow f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Example starting from a joint pdf: first try!

Example 1.1

Let $f_{X,Y}(x, y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

1. Find $\mathbb{P}(2 < X < 10 | Y = 4)$

$$P(2 < X < 10 \mid Y = 4) = \frac{P(2 < X < 10 \ \& \ Y = 4)}{P(Y = 4)}$$

prob = 0

$f_Y(y)$

What is a conditional density?

Definition: Conditional density

The conditional density of a r.v. X given $Y = y$, is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)},$$

for $f_Y(y) > 0$

Remarks

1. It follows from the definition for the conditional density $f_{X|Y}(x|y)$, that

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y).$$

2. For a fixed value of $Y = y$, the conditional density $f_{X|Y}(x|y)$ is an actual pdf, meaning
   - $f_{X|Y}(x|y) \geq 0$ for all x and y, and
   - $\int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = 1.$

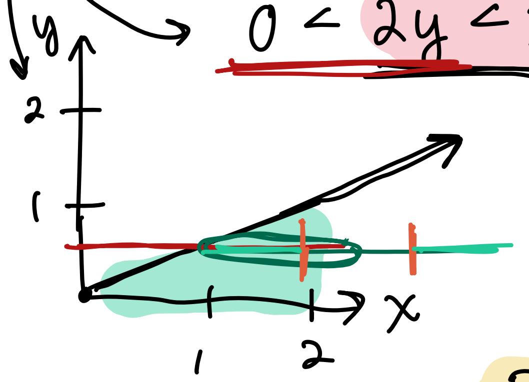
Example starting from a joint p~~m~~df: second try!

Example 1.1

Let $f_{X,Y}(x,y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$. $y = \frac{1}{2}x$

1. Find $\mathbb{P}(2 < X < 10 | Y = 4)$

$0 < 2y < x$



$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{2y}^{\infty} 5e^{-x-3y}\,dx = \int_{2y}^{\infty} 5e^{-x}e^{-3y}\,dx = (5e^{-3y})(-e^{-x})\Big|_{x=2y}^{x=\infty} = 5e^{-3y}(\cancel{-e^{-\infty}}^{\,0} + e^{-2y}) = 5e^{-3y}e^{-2y} = 5e^{-5y}$$

$y > 0$

$$= \frac{\cancel{5}e^{-x}e^{-3y}}{\cancel{5}e^{-5y}}$$

$$= e^{-x}e^{-3y+(+5y)}$$

$$= e^{-x}e^{2y} \text{ for } 0 \le y \le \frac{x}{2}$$

$$f_{X|Y}(x|y=4) = e^{-x}e^{8} \quad x \ge 8$$

$$P(2<X<10|Y=4) = \int_{2}^{10} f_{X|Y}(x|y=4)\,dx$$

$$\overbrace{\int_2^8 e^{-x}e^{8}\,dx}^{=0} + \int_8^{10} e^{-x}e^{8}\,dx$$

$$= \int_8^{10} e^{-x}e^{8}\,dx = 1 - e^{-2}$$

$$\rightarrow F_{X,Y}(x=10,\ y=4) - F_{X,Y}(\underline{x=\cancel{2},\ y=4})$$

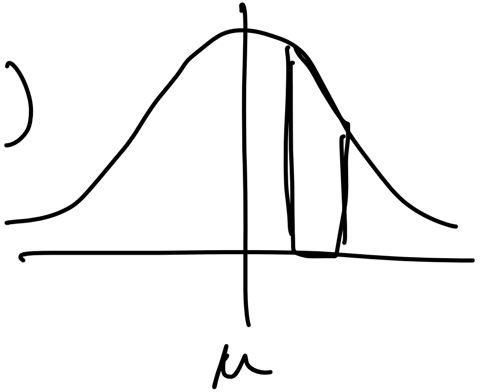
$$\downarrow\ 8$$

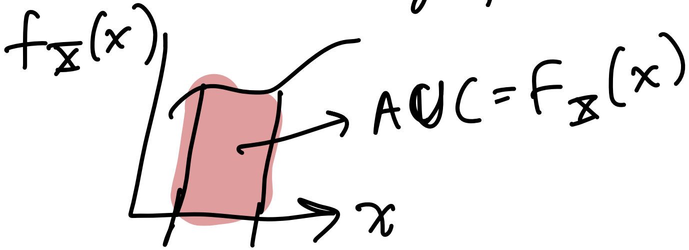
$$F_{X,Y}(x=10,\ y=4) - F_{X,Y}(x=8,\ y=4)$$

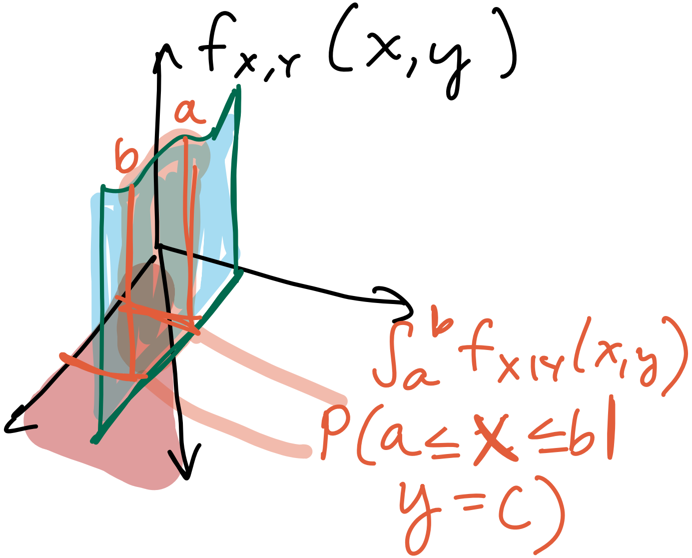
$$= \int_{8}^{10}\int_{4}^{4} f_{x,y}(x,y)\,dy\,dx$$

$$= \int_{8}^{10}\int_{4}^{4} \underline{5e^{-x}}e^{-3y}\,dy\,dx$$

$$= \int_{8}^{10}\left[5e^{-x}\left(-3e^{-3y}\right)\right]_{y=4}^{y=4} dx$$







Example starting from a joint p~~m~~df

for home

Example 1.2

Let $f_{X,Y}(x, y) = 5e^{-x-3y}$, for $0 < y < \frac{x}{2}$.

2. Find $\mathbb{P}(X > 20 | Y = 5)$

Finding probability with conditional domain and pdf

Example 2

Randomly choose a point X from the interval $[0, 1]$, and given $X = x$, randomly choose a point Y from $[0, x]$. Find $\mathbb{P}(0 < Y < \frac{1}{4})$.

Independence and conditional distributions

Question What is $f_{X|Y}(x|y)$ if X and Y are independent?

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_y(y)} = \frac{f_X(x)f_y(y)}{f_y(y)} = f_X(x)$$

- If $f_{X|Y}(x|y)$ does not depend on y (including the bounds/domain), then X and Y are independent.