Chapter 28: Expected Values of Continuous Random Variables

Meike Niederhausen and Nicky Wakim

2023-11-20

Table of contents

Learning Objectives

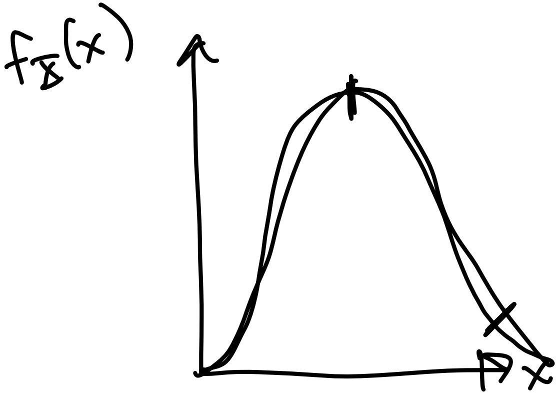
1. Calculate [redacted] mean (expected value) of a continuous RV

Expected value of a function of a continuous RV

How do we calculate expected values of discrete RVs?

For discrete RVs: weight average

$$\mathbb{E}[X] = \sum_{i=1}^{n} x_i p_X(x_i).$$

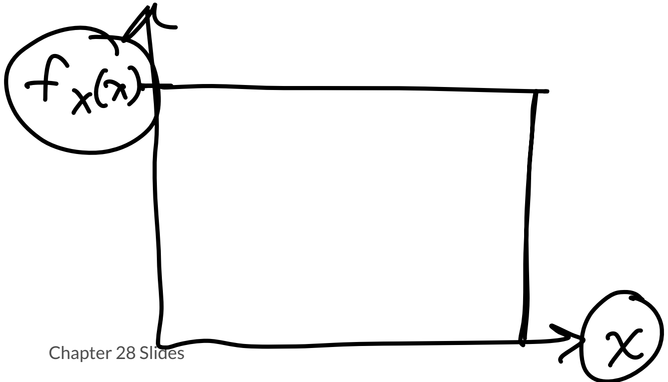


How do we calculate expected values of continuous RVs?

For continuous RVs:

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

adjust bounds to be
bounds of specific pdf ($f_X(x)$)

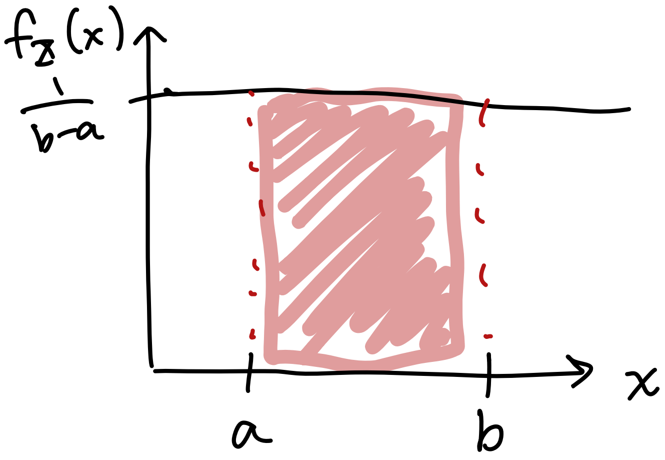


Expected Value of the Uniform Distribution

Example 1

Let $f_X(x) = \frac{1}{b-a}$, for $a \leq x \leq b$. Find $\mathbb{E}[X]$.

$$E(X) = \int_a^b x\left(\frac{1}{b-a}\right)dx = \dots = \frac{b+a}{2}$$



Expected Value of the Exponential Distribution

$X \sim \text{Exp}(\lambda)$

Example 2

Let $f_X(x) = \lambda e^{-\lambda x}$, for $x > 0$ and $\lambda > 0$. Find $\mathbb{E}[X]$.

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^{\infty} x\lambda e^{-\lambda x}\,dx$$

$$= \lambda x\left(-\frac{1}{\lambda}e^{-\lambda x}\right)\Big|_0^{\infty} - \int_0^{\infty}\left(-\frac{1}{\lambda}e^{-\lambda x}\right)\lambda\,dx$$

$x=\infty$ $\quad$ $x=0$

$$= 0 - 0 - \left(\frac{1}{\lambda}e^{-\lambda x}\right)\Big|_0^{\infty}$$

$$= -\left(0 - \left(\frac{1}{\lambda}e^{-0}\right)\right) = \frac{1}{\lambda}$$

Int by parts

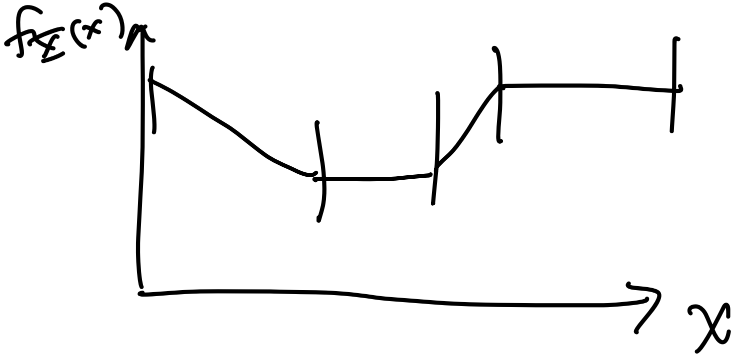
$u = \lambda x \qquad dv = e^{-\lambda x}dx$

$du = \lambda\,dx \qquad v = \frac{-1}{\lambda}e^{-\lambda x}$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$e^{-x} \xrightarrow{x\to\infty} 0$

faster than $x \to \infty$

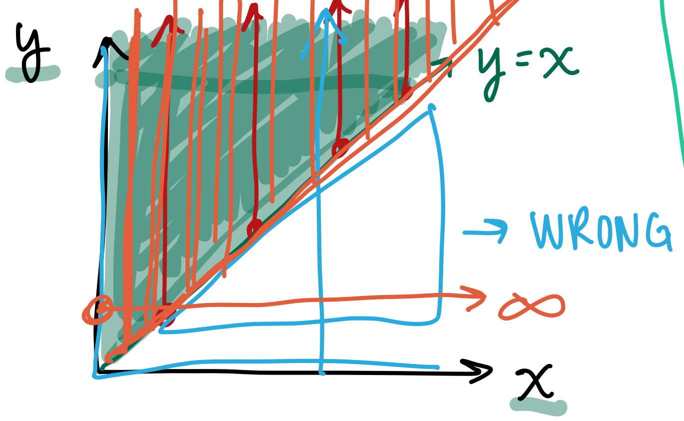


Expected value from a joint distribution

Example 3

Let $f_{X,Y}(x,y) = 2e^{-(x+y)}$, for $0 \le x \le y$. Find $\mathbb{E}[X]$.

DOMAIN:



Need to int over all y to find $f_X(x)$

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$e^{a+b} = e^a e^b$$

$$e^{-(x+y)} = e^{-x}e^{-y}$$

$$f_X(x) = \int_{y=x}^{y=\infty} f_{X,Y}(x,y)\, dy$$

$$= \boxed{\int_{y=x}^{y=\infty} 2e^{-x}e^{-y}\, dy}$$

$$= 2e^{-x}\int_x^{\infty} e^{-y}\, dy = 2e^{-x}\left[-e^{-y}\right]_{y=x}^{y=\infty}$$

$$= 2e^{-x}\Big[\underbrace{-e^{-\infty}}_{0} \underbrace{-(-e^{-x})}_{+e^{-x}}\Big] = 2e^{-2x}$$

for $x \ge 0$

$$E(X) = \int_0^{\infty} x\,\big(\underbrace{2e^{-2x}}_{\boxed{\lambda e^{-\lambda x}}}\big)\, dx$$

exp dist'n

$$= \frac{1}{\lambda} = \frac{1}{2}$$