Chapter 29: Variance of Continuous Random Variables

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Learning Objectives

1. Calculate expected value of functions of RVs
2. Calculate variance of RVs

Expected value of a function of a continuous RV

$g(x)$ is some fn of x

How do we calculate the expected value of a function of a discrete RV or joint RVs?

For discrete RVs:

$$\mathbb{E}[g(X)] = \sum_{\{\text{all } x\}} g(x) p_X(x).$$

$$\mathbb{E}[g(X, Y)] = \sum_{\{\text{all } x\}} \sum_{\{\text{all } y\}} g(x, y) p_{X,Y}(x, y).$$

How do we calculate the expected value of a function of a continuous RV or joint RVs?

For continuous RVs:

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

$$E(g(X,Y)) = \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dy\, dx$$

Expected value from a joint pdf

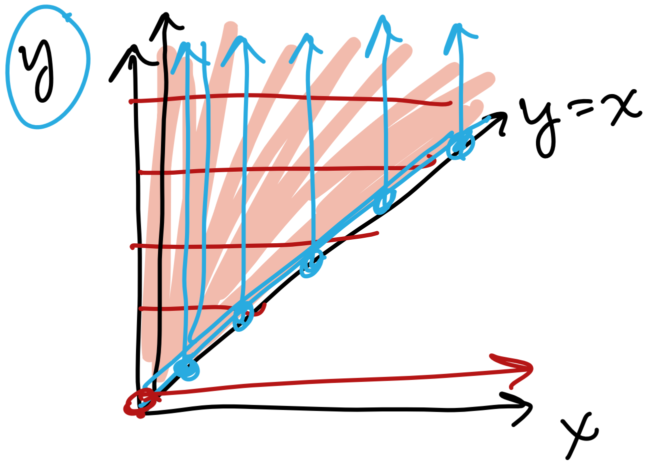
$E(X+Y) \Rightarrow g(X,Y) = x+y$

$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$] in prev notes

$g(X,Y) = x$

Example 1

Let $f_{X,Y}(x,y) = 2e^{-(x+y)}$, for $0 \le x \le y$. Find $\mathbb{E}[X]$.



$$E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_0^{\infty}\int_{y=x}^{y=\infty} x\,(2e^{-x}e^{-y})\, dy\, dx$$

$$= \int_0^{\infty}\int_0^{y} x\,(2e^{-x}e^{-y})\, dx\, dy$$

] double check that you get same answer

$$= \int_0^{\infty} x \boxed{\int_x^{\infty} 2e^{-x}e^{-y}\, dy}\, dx$$

$$= \dots = \frac{1}{2}$$

Remark on expected value of one RV from joint pdf

If you are given $f_{X,Y}(x, y)$ and want to calculate $\mathbb{E}[X]$, you have two options:

1. Find $f_X(x)$ and use it to calculate $\mathbb{E}[X]$.
2. Or, calculate $\mathbb{E}[X]$ using the joint density:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x, y) dy dx.$$

Important properties of expected values of functions of continuous RVs

Function of RV with two constants

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

Function of two RVs added

$g(X,Y) = X+Y$

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$\int\int g(x,y) f_{XY}(x,y)$

Expected value of sum of ~~independent~~ RVs pt 1

If $X_1, X_2, \ldots X_n$ are continuous RVs and $a_1, a_2, \ldots a_n$ are constants, then

$$\mathbb{E}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \mathbb{E}[X_i]$$

Expected value of multiplication of function of independent RVs

If X and Y are independent continuous RVs, and g and h are functions, then

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$

Expected value of multiplication of independent RVs

If X and Y are independent continuous RVs, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

$E[X_i \cdot X_i \cdot X_i]$

b/c $\not\perp$ $\neq E(X_i)E(X_i)E(X_i)$

Variance of continuous RVs

How do we calculate the variance of a discrete RV?

For discrete RVs:

$$\begin{aligned} Var(X) &= \mathbb{E}[(X - \mu_X)^2] \\ &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \\ &= \sum_{\{\text{all } x\}} (x - \mu_x)^2 p_X(x) \end{aligned}$$

How do we calculate the variance of a continuous RV?

For continuous RVs:

$$\begin{aligned} Var(X) &= E[(X - \mu_X)^2] \\ &= E[(X - E(X))^2] \\ &= E(X^2) - [E(X)]^2 \\ &= \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\,dx \end{aligned}$$

Variance of an Uniform distribution

Example 2

Let $f_X(x) = \frac{1}{b-a}$, for $a \leq x \leq b$. Find $Var[X]$.

Variance of exponential distribution

Example 3

Let $f_X(x) = \lambda e^{-\lambda x}$, for $x > 0$ and $\lambda > 0$. Find $Var[X]$.

Important properties of variances of continuous RVs

function of RV with two constants

$$Var[aX + b] = a^2 Var[X]$$

Variance of sum of independent RVs pt 1

If $X_1, X_2, \ldots X_n$ are independent continuous RVs and $a_1, a_2, \ldots a_n$ are constants, then

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

Variance of sum of independent RVs pt 2

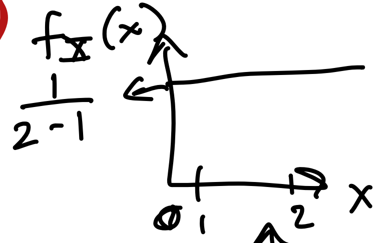
If $X_1, X_2, \ldots X_n$ are independent continuous RVs, then

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i)$$

Find the mean and sd from word problem

$[E(X_i)]^3 \to E(X_i^3)$

$\hookrightarrow X_i \not\perp X_j$



Example 4

A machine manufactures cubes with a side length that varies uniformly from 1 to 2 inches. Assume the sides of the base and height are equal. The cost to make a cube is 10 ¢ per cubic inch, and 5 ¢ ~~cents~~ for the general cost per cube. Find the mean and standard deviation of the cost to make 10 cubes.

Model cost of 10 cubes:

Let C = cost of 10 cubes

C_i = cost of ith cube for $i = 1, \dots, 10$

X_i = length cube sides for cube i

$C = \sum_{i=1}^{10} C_i \qquad C_i = 5 + 10(X_i^3) \qquad X_i \sim U[1,2]$

$$E(C) = E\left[\sum_{i=1}^{10} (5 + 10(X_i)^3)\right]$$

$$= \sum_{i=1}^{10} E[5 + 10X_i^3] = \sum_{i=1}^{10} (5 + 10E(X_i^3))$$

$$f_X(x) = \frac{1}{b-a}$$

$$E(X_i^3) = \int_1^2 x^3 \frac{1}{(2-1)} dx$$

$$= \int_1^2 x^3 dx = \frac{1}{4} x^4 \Big|_1^2$$

$$= \frac{2^4 - 1^4}{4} = 15/4 = 3.75$$

$$= \sum_{i=1}^{10} (5 + 10(15/4)) = \sum_{i=1}^{10} 42.5$$

$\neq E(X_i)E(X_i)E(X_i)$

$(1.5)^3 = 3.375$

$$= 10 \cdot 42.5 = 425¢ = \$4.25$$

$$SD(C) = \sqrt{Var(C)}$$

$$Var(C) = Var\left[\sum_{i=1}^{10}(5 + 10X_i^3)\right] = \sum_{i=1}^{10} Var(\underbrace{5 + 10X_i^3}_{C_i})$$

$Var(Y + b) = Var(Y)$

constant b

$Var(\Sigma) = \Sigma\, var$

$E(X+Y) = E(X) + E(Y)$

$X \not\perp Y$

$Var(X+Y) \neq Var(X) + Var(Y)$

refers to each cube's side

$X_i \perp X_j$, $i \neq j$

$C_i \perp C_j$, $i \neq j$

X_i's w/in each 3 sides not ind

$$= \sum_{i=1}^{10} Var(10X_i^3) = \sum_{i=1}^{10} 10^2 Var(X_i^3) = 100 \sum_{i=1}^{10} Var(X_i^3)$$

$Var(aX) = a^2 Var(X)$

$$= 100 \cdot 10 \cdot Var(X_1^3)$$

$$= 1000\, Var(X_1^3)$$

$X_i \sim U[1,2]$

identically

$Var(Y) = E(Y^2) - [E(Y)]^2$

$$Var(X_1^3) = \underbrace{E((X_1^3)^2)}_{} - (\underbrace{E[X_1^3]}_{15/4})^2$$

$$E[X_1^6] = \int_{-\infty}^{\infty} x^6 f_X(x)\,dx$$

$$= \int_1^2 x^6 \frac{1}{2-1}\,dx = \int_1^2 x^6\,dx$$

$$= \frac{1}{7}x^7 \Big|_{x=1}^{x=2} = \frac{1}{7}(2)^7 - \frac{1}{7}(1)^7$$

$$= \frac{128}{7} - \frac{1}{7} = \frac{127}{7}$$

$$E(X_1^6) - (E[X_1^3])^2$$

$$= \frac{127}{7} - \left(\frac{15}{4}\right)^2 = 4.0803$$

$$Var(C) = 1000 \cdot Var(X_1^3) = 1000 \cdot 4.0803$$

$$= 4,080.357$$

$$SD(C) = \sqrt{Var(C)} = \sqrt{4080.357} = 63.8776\ldots$$

$$SD(C) = 64¢$$