Chapter 32: Exponential Random Variables

Meike Niederhausen and Nicky Wakim

2023-11-27

Table of contents

Learning Objectives

1. Identify the variable and the parameters in a story, and state in English what the variable and its parameters mean.
2. Use the formulas for the density, CDF, expected value, and variance to answer questions and find probabilities.

Properties of exponential RVs

- **Scenario:** Modeling the time until the next (first) event/success
- Continuous analog to the geometric distribution!
- Shorthand: $X \sim \text{Exp}(\lambda)$

λ = avg rate of successes

OR

λ = avg # of arrivals/successes per time period

$$f_X(x) = \lambda e^{-\lambda x} \text{ for } x > 0, \lambda > 0$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$$

$$E(X) = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$

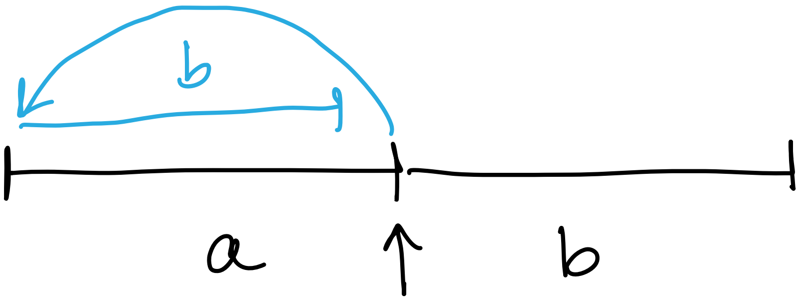
Memoryless Property

If $b > 0$,

$$P(X > a + b | X > a) = P(X > b)$$

no successes
in a time

- **This can be interpreted as:**
  - If you have waited a seconds (or any other measure of time) without a success
  - Then the probability that you have to wait b more seconds is the same as as the probability of waiting b seconds initially.



Identifying exponential RV from word problems

- Look for time between events/successes
- Look for a rate of the events over time period (avg rate)
- How does it differ from the geometric distribution?
  - Geometric is *number of trials* until first success
  - Exponential is *time* until first success
- Relation to the Poisson distribution?
  - When the time between arrivals is exponential, the number of arrivals in a fixed time interval is Poisson with the mean λ

time int.

count successes in time interval

Helpful R code

Let's say we're sitting at the bus stop, measuring the time until our bus arrives. We know the bus comes every 10 minutes on average.

1 bus / 10 min.

- If we want to know the probability that the bus arrives in the next 5 minutes: $P(X \leq 5) = 0.393$

CDF

```
pexp(q = 5, rate = 1/10)
```

0

```
[1] 0.3934693
```

- If we want to know the time, say t, where the probability of the bus arriving at t or earlier is 0.35: $P(X \leq t) = 0.35$

```
qexp(p = 0.35, rate = 1/10)
```

```
[1] 4.307829
```

- If we want to know the probability that the bus arrives between 3 and 5 minutes: $P(3 \leq X \leq 5)$

```
pexp(q = 5, rate = 1/10) - pexp(q = 3, rate = 1/10)
```

```
[1] 0.1342876
```

- If we want to sample 20 bus arrival times from the distribution:

```
rexp(n = 20, rate = 1/10)
```

```
 [1] 30.0816505 21.7194972 16.3610530  8.3069168  6.5553312  1.3089050
 [7]  9.6093683 10.1363398  0.6275717 14.8675929  4.3278190  6.7367828
[13] 31.0445867  4.0943354 20.5852011  0.6137783 11.3663773  4.4192161
[19]  8.8711282  7.0947632
```

Transformation of independent exponential RVs

Example 1

Let $X_i \sim \text{Exp}(\lambda_i)$ be independent RVs, for $i = 1 \ldots n$. Find the pdf for the first of the arrival times.

Let M = first arrival time

$M = \min(X_1, X_2, \ldots, X_n)$

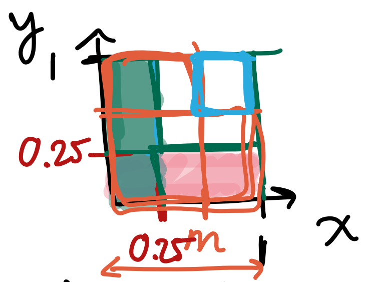
w/ $X_i \sim \text{Exp}(\lambda_i)$

Transformation:

① Create our CDF

② derivative of CDF to get pdf

$F_M(m) = P(M \leq m) = P(\min(X_1, X_2, \ldots, X_n) \leq m)$



$= 1 - P(\min(X_1, X_2, \ldots X_n) \geq m)$

$= 1 - P(X_1 \geq m, X_2 \geq m, \ldots, X_n \geq m)$

$= 1 - P(X_1 \geq m) P(X_2 \geq m) \cdots P(X_n \geq m)$

$= 1 - (1 - F_{X_1}(m))(1 - F_{X_2}(m)) \cdots (1 - F_{X_n}(m))$

$= 1 - [\cancel{1} - (\cancel{1} + e^{-\lambda_1 m})][\cancel{1} - (\cancel{1} + e^{-\lambda_2 m})] \cdots [\cancel{1} - (\cancel{1} + e^{-\lambda_n m})]$

$$= 1 - e^{-\lambda_1 m} e^{-\lambda_2 m} \cdots e^{-\lambda_n m}$$

$e^a e^b = e^{a+b}$

$$= 1 - e^{-\lambda_1 m - \lambda_2 m - \cdots - \lambda_n m}$$

$$= 1 - e^{-m(\lambda_1 + \lambda_2 + \cdots \lambda_n)}$$

$$F_M(m) = 1 - e^{-m \sum_{i=1}^{n} \lambda_i}$$

$$f_M(m) = \frac{d}{dm} F_M(m) = \frac{d}{dm}\left(1 - e^{-m \sum_{i=1}^{n} \lambda_i}\right)$$

$$= 0 - \left(-\sum_{i=1}^{n} \lambda_i\right) e^{-m \sum_{i=1}^{n} \lambda_i}$$

$\rightarrow a = -\sum \lambda_i$

$$\boxed{\frac{d}{dm} e^{am} = a e^{am}}$$

$$f_M(m) = \left(\sum_{i=1}^{n} \lambda_i\right) e^{-m \sum_{i=1}^{n} \lambda_i}$$

$m > 0$, $\lambda_i > 0$ for $i = 1, \ldots, n$

$$M \sim Exp\left(\sum_{i=1}^{n} \lambda_i\right)$$

$\lambda e^{-\lambda m}$

$\hookrightarrow \sum \lambda_i$