Chapter 33: Gamma Random Variables

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Learning Objectives

1. Identify the variable and the parameters in a story, and state in English what the variable and its parameters mean.

Properties of gamma RVs

- **Scenario:** Modeling the time until the r^{th} event.
- Continuous analog to the Negative Binomial distribution
- Shorthand: $X \sim \text{Gamma}(r, \lambda)$ or $\Gamma(r, \lambda)$

λ is rate

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)} x^{r-1} \lambda e^{-\lambda x} \text{ for } x > 0, \lambda > 0, \Gamma(r) = (r-1)!$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda x} \sum_{j=0}^{r-1} \frac{(\lambda x)^j}{j!} & x \geq 0 \end{cases}$$

$$E(X) = \frac{r}{\lambda}, \quad \text{Var}(X) = \frac{r}{\lambda^2}$$

Common to see $\alpha = r$ and $\beta = \lambda$

Identifying gamma RV from word problems

- Gamma distribution with $r = 1$ is same as exponential
  - Just like Negative Binomial with $r = 1$ is same as the geometric distribution
- Similar to exponential
  - Look for time between or until events/successes X
    - BUT now we are measuring time until more than 1 success
  - Look for a rate of the events over time period

Helpful R code

Let's say we're sitting at the bus stop, measuring the time until 4 buses arrive. We know the bus comes every 10 minutes on average.

$r = 4$, rate: $\frac{1 \text{ bus}}{10 \text{ min}}$, $\lambda = \frac{1}{10}$

$X \sim Gamma(r = 4, \lambda = \frac{1}{10})$

- If we want to know the probability that the 4 buses arrive in the next 50 minutes: $P(X \leq 50)$

```
pgamma(q = 50, rate = 1/10, shape = 4)
```

```
[1] 0.7349741
```

```
pgamma(q = 50, scale = 10, shape = 4)
```

```
[1] 0.7349741
```

$\to 1/\lambda = \frac{1}{rate}$

- If we want to know the time, say t, where the probability of the 4 buses arriving at t or earlier is 0.35: $P(X \leq t) = 0.35$

```
qgamma(p = 0.35, rate = 1/10, shape = 4)
```

$F_X(t) = 0.35$

```
[1] 29.87645
```

- If we want to know the probability that the 4 buses arrives between 30 and 50 minutes: $P(30 \leq X \leq 50)$

```
pgamma(q = 50, scale = 10, shape = 4) - pgamma(q = 30, scale = 10, shape = 4)
```

```
[1] 0.382206
```

- If we want to sample 20 arrival times for the 4 buses:

```
rgamma(n = 20, scale = 10, shape = 4)
```

```
 [1] 34.34812 61.47293 24.72448 70.00885 15.05334 40.56682 47.79459 30.41611
 [9] 51.68054 14.14803 23.84225 14.68795 44.53299 37.95460 54.29088 19.53350
[17] 59.67360 19.99544 63.29344 19.73288
```

Remarks

- The parameter r in a Gamma(r,λ) distribution does NOT need to be a positive integer
  - r is usually a positive integer
- When r is a positive integer, the distribution is sometimes called an Erlang(r,λ) distribution

- When r is any positive real number, we have a general gamma distribution that is usually instead parameterized by $\alpha > 0$ and $\beta > 0$, where:
  - $\alpha = \text{shape parameter}$: same as k, the total number of events we must witness
    - In R code example: 4 buses to wait for
  - $\beta = \text{scale parameter}$: same as λ, the rate parameter
    - In R code example: 1 bus per 10 minutes (1/10)

Sending money orders

Example 1

On average, someone sends a money order once per 15 minutes. What is the probability someone sends 10 money orders in less than 3 hours?

Gamma $\lambda = \frac{1 \text{ order}}{15 \text{ min}} \times \frac{60 \text{ min}}{\text{hr}} = \frac{60}{15} \frac{\text{order}}{\text{hr}}$

$r = 10$

X = time until 10 money order sent

$X \sim \text{Gamma}(r = 10, \lambda = \frac{60}{15})$

$P(X < 3) = 0.7576$

used R to calc

Additional Resource

- Another helpful site with R code: https://rpubs.com/mpfoley73/459051